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EXTREME COPOSITIVE QUADRATIC FORMS, II

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A real quadratic form $Q = Q(x_1, \cdots, x_n)$ is called copositive if $Q(x_1, \cdots, x_n) \geqq 0$ whenever $x_1, \cdots, x_n \geqq 0$. If we associate each quadratic form $Q = \sum q_{ij}x_ix_j$ $q_{ij} = q_{ji}$ $(i, j = 1, \cdots, n)$ with a point

$$(q_{11}, \cdots, q_{nn}, \sqrt{2}\,q_{12}, \cdots, \sqrt{2}\,q_{n-1,n})$$

of Euclidean $n(n+1)/2$ space, then the copositive forms constitute a closed convex cone in this space. We are concerned with the extreme points of this cone. That is, with those copositive quadratic forms Q for which $Q = Q_1 + Q_2$ (with Q_1, Q_2 copositive) implies $Q_1 = aQ, Q_2 = (1-a)Q, 0 \leqq a \leqq 1$. In this paper we limit ourselves almost entirely to 5-variable forms and announce the discovery of an hitherto unknown class of extreme copositive quadratic forms in 5 variables. In view of the known extension process whereby extreme copositive quadratic forms in n variables may be used to generate extreme forms in n' variables for any $n' > n > 2$, this new class of forms thus provides new extreme copositive forms in any number of variables $n' \geqq 5$.

Copositive quadratic forms arise in the theory of inequalities and also in the study of block designs. The paper of Diananda [2] provides the connection with inequalities while the paper of Hall and Newman [3] outlines the application of copositive quadratic forms to block designs.

2. **Preliminaries.** As indicated above, a real quadratic form $Q = Q(x_1, \cdots, x_n)$ is called copositive if $Q(x_1, \cdots, x_n) \geqq 0$ whenever $x_1, \cdots, x_n \geqq 0$. Thus any positive semi-definite quadratic form is copositive. Further, any quadratic form all of whose coefficients are nonnegative in clearly copositive. Denoting these classes of forms by S and P respectively, we see that any quadratic form expressible as a sum of elements of P and S is necessarily copositive. In fact, Diananda [2, Th. 2] has shown that all copositive quadratic forms in $n \leqq 4$ variables are of this type (i.e., $Q \in P + S$ if Q is copositive and $n \leqq 4$). On the other hand, A. Horn [3] has constructed an extreme copositive quadratic form in 5 variables which does *not* belong to $P + S$. The extreme copositive quadratic forms belonging to $P + S$ have been determined by Hall and Newman [3, Th. 3.2]; thus we can restrict our attention to those outside of $P + S$ whenever it is desirable to do so. (Complete details of this theorem of Hall and Newman are given in the first paragraph of § 4 below.)

If $Q(x_1, \cdots, x_n)$ is an extreme copositive form so is $Q(p_1x_1, \cdots, p_nx_n)$