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A HELLINGER INTEGRAL REPRESENTATION FOR BOUNDED LINEAR FUNCTIONALS

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The function space considered is that consisting of the complex-valued, quasicontinuous functions on a real interval $[a, b]$, anchored at a, and having the LUB norm. It is shown that each bounded linear functional on this Banach space has a Hellinger integral representation. A formula for the norm of the functional is given in terms of the integrating functions involved in its representation. A new existence criterion for the Hellinger integral is uncovered on the way to the representation theorem.

2. Definitions. In this section certain definitions and notational conventions are adopted for use in the succeeding sections. Throughout the paper, $[a, b]$ will denote a given interval and the word function will mean map from $[a, b]$ into the complex numbers.

DEFINITION 2.1. If c is any number in $(a, b]$, then R_c denotes a function such that $R_c(t) = 0$ if t is in $[a, c)$ and $R_c(t) = 1$ if $c \leqq t \leqq b$. If c is in $[a, b)$, then L_c denotes a function such that $L_c(t) = 0$ if $a \leqq t \leqq c$ and $L_c(t) = 1$ is t is in $(c, b]$. The functions L_c and R_c are called unit step functions. A linear combination of unit step functions is called a step function. Notice that each step function vanishes at a.

DEFINITION 2.2. We now specify the function space, $Q_0[a, b]$, which plays the central role. Its elements are the quasicontinuous functions anchored at a and they may be defined in two ways. First, $Q_0[a, b]$ is the set of all functions which vanish at a and which have a limit from the right at each t in $[a, b)$ and a limit from the left at each t in $(a, b]$. Second, let $B[a, b]$ be the Banach space of bounded functions, with LUB norm. Then $Q_0[a, b]$ is the closure, in $B[a, b]$, of the linear space of all step functions. So $Q_0[a, b]$ is a Banach space with norm $\|x\| = LUB\,|x(t)|$ for all t in $[a, b]$. Also, each bounded linear functional on $Q_0[a, b]$ is determined by its values on the step functions, since the latter form a dense linear subspace.

For proof of the equivalence of these two formulations of $Q_0[a, b]$, see [1, Lemma 4.16].

DEFINITION 2.3. Suppose g is any subset of $[a, b]$. If x is a function, then x_g denotes a function such that $x_g(t) = x(t)$ if t is in g