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LOWER BOUNDS FOR THE EIGENVALUES OF A VIBRATING STRING WHOSE DENSITY SATISFIES A LIPSCHITZ CONDITION

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If a string has a density given by a nonnegative integrable function ρ defined on the interval $[0, a]$ and is fixed at its end points under unit tension, then the natural frequencies of vibration of the string are determined by the eigenvalues of the differential system

$$u'' + \lambda\rho(x)u = 0\ ,\ u(0) = u(a) = 0\ . \tag{1}$$

As is well known, the eigenvalues of (1) form a positive strictly increasing sequence of numbers which depend on the density $\rho(x)$. We denote them accordingly by

$$0 < \lambda_1[\rho] < \lambda_2[\rho] < \cdots < \lambda_n[\rho] < \cdots .$$

In this paper we find lower bounds for these eigenvalues when the density ρ satisfies a Lipschitz condition with Lipschitz constant H and $\int_0^a \rho dx = M$. The bounds will be in terms of M and H.

Specifically, if $E(H, M)$ is the family of functions

$$\left\{\rho : \rho \in L(H) \text{ and } \int_0^a \rho(x)dx = M\right\}, \tag{2}$$

where

$$L(H) = \{\rho : |\rho(x_1) - \rho(x_2)| \leqq H|x_1 - x_2|;\ x_1, x_2 \in [0, a]\}\ ,$$

we find a unique function $\rho_0 \in E(H, M)$ for each $\lambda_n[\rho]$ such that

$$\lambda_n[\rho_0] = \min \lambda_n[\rho]$$

where the minimum is taken over all functions $\rho \in E(H, M)$.

Our results will be expressed in terms of the fundamental pair of solutions U_1 and U_2 of the Airy equation

$$\frac{d^2U}{ds^2} + sU = 0 \tag{3}$$

where $U_1(0) = 1$, $U_1'(0) = 0$ and $U_2(0) = 0$, $U_2'(0) = 1$. These functions are tabulated in [7]. The main conclusion is that

$$\min_E \lambda_n[\rho] = \left[nt_1\left(\frac{Ha^2}{nM}\right)\right]^3 \Big/ Ha^3$$