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INVARIANT SUBSPACES OF POLYNOMIALLY COMPACT OPERATORS ON BANACH SPACE

ALLEN R. BERNSTEIN

This paper contains a proof of the following:

MAIN THEOREM. Let T be a bounded linear operator on an infinite-dimensional Banach space B over the complex numbers. Suppose there exists a polynomial $p(\lambda) \neq 0$ with complex coefficients such that $p(T)$ is compact (completely continuous). Then T leaves invariant at least one closed linear subspace of B other than $\{0\}$ or B.

Since in most of the common nonreflexive Banach spaces (e.g. l_1, $C[0, 1]$, AP, etc.) weakly compact operators have compact squares (cf. [7], pp. 511 and 580), one can conclude in particular from the above theorem that these operators have proper invariant subspaces.

The proof of the main theorem is carried out within the framework of A. Robinson's Theory of Nonstandard Analysis and follows lines similar to the proof presented in [3] for the special case where B is a Hilbert space, which settled a question raised by P. Halmos and K. Smith [9], That proof made strong use of the fact that in a separable Hilbert space it is possible to choose a countable orthonormal basis. In a general separable Banach space, of course, one does not even know whether there is a basis, much less an "orthonormal" basis. However this difficulty may be overcome by the introduction of *metric projections* to take the place of projections in Hilbert space, as was done by Aronszajn and Smith in their proof of the existence of invariant subspaces for compact operators [1], and by the introduction of a *semi-basis* to take the place of the orthonormal basis in Hilbert space.

The proof in [3] was carried out within the framework of a nonstandard model of the real numbers. This was possible because the elements of a separable Hilbert space may be represented as sequences of complex numbers which in turn may be defined as ordered pairs of real numbers. However, in the case of a general Banach space no such convenient representation is apparent so that a more general version of nonstandard analysis is necessary. The remainder of this section is an introduction to such a version drawn essentially from the work of A. Robinson ([11]-[14]).

The class T of *types* is defined inductively as follows. (i) 0 is a type; (ii) if $\tau_1, \cdots, \tau_n$ are types, $n \geqq 1$, then $(\tau_1, \cdots, \tau_n)$ is a type; (iii) T is the smallest class satisfying (i) and (ii).

A *higher order structure* is defined to be a generalized sequence