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ON THE WEAK LAW OF LARGE NUMBERS AND THE GENERALIZED ELEMENTARY RENEWAL THEOREM

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$\{X_n\}$ is a sequence of independent, nonnegative, random variables and $G_n(x) = P\{X_1 + \cdots + X_n \leqq x\}$. $\{a_n\}$ is a sequence of nonnegative constants such that, for some $\alpha > 0, \gamma > 0$, and function of slow growth $L(x)$,

$$\sum_1^N a_r \sim \frac{\alpha N^\gamma L(N)}{\Gamma(1+\gamma)}, \quad \text{as } N \to \infty .$$

A Generalized Elementary Renewal Theorem (GERT) gives conditions such that, for some $\mu > 0$,

$$(*) \qquad \Psi(x) = \Sigma a_r G_r(x) \sim \frac{\alpha L(x)}{\Gamma(1+\gamma)}\left(\frac{x}{\mu}\right)^\gamma, \quad \text{as } x \to \infty .$$

The Weak Low of Large Numbers (WLLN) states that $(X_1 + \cdots + X_n)/n \to \mu$, as $n \to \infty$, in probability. Theorem 1 proves that WLLN implies (*). Theorem 3 proves that (*) implies WLLN if, additionally, it is given that

(i) $\sum_1^n P\{X_j > n\varepsilon\} \to 0$ as $n \to \infty$, for every small $\varepsilon > 0$;

(ii) for some $\varepsilon > 0$, $n^{-1}\sum_1^n \int_0^{n\varepsilon} P\{X_j > x\}dx$ is a bounded function of n. Theorem 2 supposes the $\{X_n\}$ to have finite expectations and proves (*) implies WLLN if it is given that

$$\limsup_{n\to\infty} \frac{\mathscr{E}X_1 + \mathscr{E}X_2 + \cdots + \mathscr{E}X_n}{n} \leqq \mu ,$$

in which case $(\mathscr{E}X_1 + \cdots + \mathscr{E}X_n)/n$ necessarily tends to μ as $n \to \infty$. Finally, an example shows that (*) can hold while the WLLN fails to hold. Much use is made of the fact that a necessary and sufficient condition for the WLLN is that, for all small $\varepsilon > 0$,

$$\frac{1}{n}\int_0^{n\varepsilon} \sum_1^n P\{X_j > x\}dx \to \mu, \quad \text{as } n \to \infty .$$

Let $\{X_n\}$, $n = 1, 2, \cdots$, be a sequence of independent, *nonnegative*, random variables; write $F_n(x) \equiv P\{X_n \leqq x\}$; $S_n = X_1 + X_2 + \cdots + X_n$; $G_n(x) = P\{S_n \leqq x\}$; when the first moments exist, write $\mu_n = \mathscr{E}X_n$. Let $\{a_n\}$ be a sequence of nonnegative constants such that, for some constants $\alpha > 0$, $\gamma > 0$, and some function of slow growth $L(x)$,

$$(1.1) \qquad \sum_{n=1}^N a_n \sim \frac{\alpha N^\gamma L(N)}{\Gamma(1+\gamma)}, \quad \text{as } N \to \infty .[1]$$

[1] We carry the factor $\Gamma(1+\gamma)$ to simplify comparisons with Smith (1964).