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A SEMIGROUP UNION OF DISJOINT LOCALLY FINITE SUBSEMIGROUPS WHICH IS NOT LOCALLY FINITE

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The semigroup S of the title is the free semigroup F on four generators factored by the congruence generated by the set of relations $\{w^2 = w^3 \mid w \in F\}$. The following lemma is proved by examining the elements of a given congruence class of F:

LEMMA. If $x, y \in S$ and $x^2 = y^2$, then either $xy = x^2$ or $yx = x^2$.

From the Lemma it then easily follows that the (disjoint) subsemigroups $\{y \in S \mid y^2 = x^2\}$ of S are locally finite.

This note answers in the negative a question raised by Shevrin in [2].

THEOREM. *There exists a semigroup S with disjoint locally finite subsemigroups S_e such that $S = \cup S_e$ and S is not locally finite.*

Let F be the free semigroup with identity on four generators. Let $\sim$ denote the smallest congruence on F containing the set $\{(x^2, x^3) \mid x \in F\}$. That is, for $w, w' \in F$, $w \sim w'$ if and only if a finite sequence of "transitions", of either of the types $ab^2c \to ab^3c$ or $ab^3c \to ab^2c$, transforms w into w'.

The equivalence classes of F with respect to $\sim$ are taken as the elements of S, and multiplication in S is defined in the natural way.

There is given in [1] a sequence on four symbols in which no block of length k is immediately repeated, for any k. Thus the left initial segments of this sequence give elements of F containing no squares. Since no transition of the form $ab^2c \to ab^3c$ or $ab^3c \to ab^2c$ can be applied to an element of F containing no squares, the equivalence classes containing these elements consist of precisely one element each; thus the semigroup S is infinite, and hence not locally finite.

In what follows, the symbols $\alpha, \alpha_1, \alpha_2, \cdots$ refer to transformations (on elements of F) of the form

$$ab \to ayb\ , \quad \text{where } a \sim ay\ , \quad \text{and} \quad a, b, y \in F\ .$$

The symbols $\beta, \beta_1, \beta_2, \cdots$ refer to transformations of the type

$$axb \to ab\ , \quad \text{where } a \sim ax\ , \quad \text{and} \quad a, b, x \in F\ .$$

Note that $ab^2c \to ab^3c$ is an α, and $ab^3c \to ab^2c$ is a β.