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SOME MAPPING PROPERTIES OF THE GROUP ALGEBRAS OF A COMPACT GROUP

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A number of equivalent conditions are given under which certain representations of $M(G)$ and $L^1(G)$ have closed range. A faithful representation with closed range implies the finiteness of G. Weakly compact operators on $L^1(G)$ commuting with right translations are classified as left convolutions by functions in $L^1(G)$.

In recent years various authors have shown that the same techniques which have proved so successful in developing the theory of locally compact abelian groups can be applied, with surprisingly small modifications, to nonabelian groups as well. The purpose of this paper is to carry on this theme with special emphasis on compact groups. Toward this end we shall emphasize the use of functional analysis methods and representation theory. In fact, the main technical result of the paper doesn't concern groups at all, though its foremost application, Theorem 2, gives some equivalent conditions under which certain representations of the group algebras of a compact group have closed range. The other main result of the paper is Theorem 4, which characterizes the weakly compact operators in the group algebra $L^1(G)$ [G compact] which commute with right translations as left convolution by some element of $L^1(G)$.

1. Notation and preliminaries. Let G be a compact Hausdorff group. Following Dixmier [2] we denote the set of equivalence classes of irreducible unitary representations of G by $\hat{G}$. We shall assume that one representation is chosen from each equivalence class, and we shall denote this collection by $\hat{G}$ as well. If U_r is in $\hat{G}$, then by the Peter-Weyl Theorem, U_r is a finite-dimensional representation, say of dimension d_r, on the Hilbert space H_r. For each such r let $B(H_r)$ be the full matrix algebra over H_r with the usual operator norm. Let $TC(H_r)$ be the full matrix algebra over H_r with the trace norm [i.e. the trace class of H_r]. Let M be the bounded direct product of all the $B(H_r)$. That is, if Δ is an index set for $\hat{G}$, then M is that subset of the Cartesian product of $\{B(H_r)\}_{r\in\Delta}$ for which $\{\|a_r\|\}_{r\in\Delta}$ is bounded, where $\{a_r\}\in M$. Under coordinatewise operations and norm $\|a\| = \sup_{r\in\Delta}\|a_r\|$, for $a=\{a_r\}\in M$, it is well-known that M is a W^*-algebra, i.e. a von Neumann algebra. Let N be the subalgebra of M defined by the condition: $a\in N$ if and only if for any $\alpha>0$, $\{r\in\Delta: \|a_r\|>\alpha\}$ is a finite set. It is clear that N is a norm-closed subalgebra of M and hence a B^*-algebra. Let F