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A RADON-NIKODYM THEOREM FOR FINITELY ADDITIVE SET FUNCTIONS

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Suppose that Σ is a field of subsets of the set S, and suppose that μ and γ are complex-valued finitely additive set functions defined on Σ. Assume that μ is bounded and γ is finite and absolutely continuous with respect to μ. (A word of warning is in order here. The statement "γ is absolutely continuous with respect to μ" is often interpreted as "$\mu(E) = 0$ implies $\gamma(E) = 0$". This is not the meaning used here. Our definition is "for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|\mu(E)| < \delta$ implies $|\gamma(E)| < \varepsilon$." Unless μ is bounded and countably additive, the two definitions are not equivalent.)

THEOREM 1. There exists a sequence $\{f_n\}$ of μ-simple functions on S, such that

1. $$\lim_{n\to\infty} \int_E f_n(s)\mu(ds) = \gamma(E) ,$$

uniformly for $E \in \Sigma$

2. $$\lim_{n,m\to\infty} \int_S |f_n(s) - f_m(s)|\, v(\mu, ds) = 0 ,$$

where $v(\mu)$ is the total variation of μ.

Theorem 1 is established by a pure existence proof, and gives no indication of how to find f_n. A more constructive result is given below.

A partition of S is a finite collection of sets E_i belonging to Σ, such that S is the disjoint union of the E_i, and such that $\mu(E_i) \neq 0$, $i = 1, \cdots, n$.

The set $\mathscr{P}$ of partitions may be directed by refinement, that is, by the following partial order: $P_1 < P_2$ if for every $E \in P_1$ there exist $F_1, \cdots, F_r \in P_2$ (r may depend on E) such that E and $\bigcup_{i=1}^r F_i$ differ by a μ-null set.

If P is a partition of S, define the μ-simple function f_P^γ to be $\sum_{E\in P} (\gamma(E)/\mu(E))\chi_E$, where χ_E is the characteristic function of E.

THEOREM 2. If μ is positive, then

$$\lim_{P\in\mathscr{P}} \int_E f_P^\gamma(s)\mu(ds) = \gamma(E) ,$$

uniformly for $E \in \Sigma$, where $\mathscr{P}$ is directed as explained above.

(The notation throughout is essentially that of [2].) For positive μ, Theorem 1 reduces to Bochner's Radon-Nikodym Theorem. See [1].