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UNSTABLE POINTS IN THE HYPERSPACE OF CONNECTED SUBSETS

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A topological property which has proved useful is that of possessing an unstable point. It is thus interesting to see which topological spaces consist entirely of unstable points. The purpose of this paper is to describe a class of such spaces. This is done in the

THEOREM. If X is a finite simplicial complex then the hyperspace $C(X)$ consists entirely of unstable points if and only if X has no free 1-simplex.

The proof given here is for the case where X is connected —the more general theorem follows obviously from this case.

2. **Definitions and remarks.** A point p in a space Z is called *unstable* if for each open neighborhood U of p there is a homotopy $h_t: Z \to Z$ such that $h_0 = 1$, $p \notin h_1(Z)$, and for all t, $h_t|_{Z \setminus U} = 1$ and $h_t(U) \subset U$. (Here 1 denotes the identity mapping on Z.) A point which is not unstable is called *stable*. (Synonyms for "unstable" are "labile" and "homotopically labil"; see, for example, [2].)

If (X, d) is a compact metric space we denote by 2^X the collection of all nonempty, closed subsets of X. When furnished with the Hausdorff metric [3, p. 167] 2^X is called the *hyperspace* of closed subsets of X. The Hausdorff metric can be defined as follows: for $x \in X$ and $E \subset X$ we define $\operatorname{dist}(x, E) = \inf\{d(x, y) \mid y \in E\}$. For $\varepsilon > 0$ we define $V_\varepsilon(E) = \{x \in X \mid \operatorname{dist}(x, E) < \varepsilon\}$. Then the Hausdorff metric ρ on 2^X is given by $\rho(E, E') = \inf\{\varepsilon > 0 \mid E \subset V_\varepsilon(E') \text{ and } E' \subset V_\varepsilon(E)\}$. Note that in order to show $\rho(E, E') < \varepsilon$ it suffices to show the following: $\operatorname{dist}(x, E') < \varepsilon$ for each $x \in E$ and $\operatorname{dist}(x', E) < \varepsilon$ for each $x' \in E'$. The subspace of $(2^X, \rho)$ consisting of the nonempty, closed, connected subsets of X is denoted by $C(X)$. (We still use the metric ρ on $C(X)$.) It is well known that $C(X)$ and 2^X are compact.

A function $f: (X, d) \to (X, d)$ is called an ε-mapping of X if $d(x, f(x)) < \varepsilon$ for each x in X.

Finite simplicial complex denotes a geometric realization of an abstract finite simplicial complex, and has the Euclidean topology.

A 1-simplex is called *free* if it is not a proper face of some other simplex in the complex.

If (X, d) is a metric space then d is said to be a *convex metric* [1, p. 1101] if for each pair of points a, b in X there is a point z in X such that $d(a, z) = d(z, b) = 1/2\, d(a, b)$. In case (X, d) is complete this is equivalent to: for each pair of points a, b in X there is a set