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INTERPOSITION AND APPROXIMATION

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Let $\mathscr{B}(X)$ be the space of all bounded real-valued functions on a set X, with the norm $\|f\| = \sup\{|f(x)|: x \in X\}$, and let K be any nonempty subset of $\mathscr{B}(X)$. The question whether an element f of $\mathscr{B}(X)$ has a best approximation g in K (such that $\|f - g\| = \delta(f) = \inf\{\|f - h\|: h \in K\}$) can be formulated as the problem of interposing a function g in K between two functions, $L(\cdot, f)$ and $U(\cdot, f)$, which are constructed out of K by certain lattice operations. If K is closed with respect to these lattice operations, or has a certain interposition property, the best approximation will always exist.

For example, X might be a bounded subset of a Banach space E and K might be the set of restrictions to X of the continuous linear functionals in E^* [2, 6]. $U(\cdot, f)$ is then constructed in two stages: first the suprema of bounded subsets of K are formed, and then $U(\cdot, f)$ is obtained as a decreasing sequential limit of such suprema. In two other typical cases, K consists of the bounded continuous functions on a paracompact space [5], or the distance decreasing functions on a metric space. These two share the property of *translational invariance*:

$$\text{(1)} \qquad \text{if } f \in K \text{ and } c \text{ is a constant, then } (f + c) \in K ,$$

which permits $U(\cdot, f)$ to be constructed by forming suprema alone, without the intervention of decreasing sequential limits. In the last of these sample cases, it actually turns out that $U(\cdot, f)$ is itself in K, and is thus the *largest* of the best approximators to f in K.

1. Mere existence. For every $\rho > 0$, there is a $g \in K$ such that $\|f - g\| < \delta(f) + \rho$, or in other words $f - \delta(f) - \rho < g < f + \delta(f) + \rho$. Therefore, $U_\rho(x, f) = \sup\{g(x): g \in K, g \leqq f + \delta(f) + \rho\}$ lies between $f - \delta(f) - \rho$ and $f + \delta(f) + \rho$, and dominates

$$L_\rho(x, f) = \inf\{g(x): g \in K, g \geqq f - \delta(f) - \rho\} .$$

L_ρ and U_ρ are, respectively, monotonically increasing and decreasing functions of ρ, so that

$$\begin{aligned} f - \delta(f) \leqq L(\cdot, f) = \lim_{n\to\infty} L_{1/n}(\cdot, f) &\leqq \lim_{n\to\infty} U_{1/n}(\cdot, f) \\ &= U(\cdot, f) \leqq f + \delta(f) . \end{aligned}$$