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A NOTE ON FUNCTIONS WHICH OPERATE

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Let $\mathfrak{A}$, $\mathscr{B}$ denote two families of functions $a, b: X \to Y$. A function $F: Z \subseteq Y \to Y$ is said to operate in $(\mathfrak{A}, \mathscr{B})$ provided that for each $a \in \mathfrak{A}$ with range $(a) \subseteq Z$ we have $F(a) \in \mathscr{B}$. Let G denote a locally compact Abelian group. In this paper we characterize the functions which operate in two cases:

(i) $\mathfrak{A} = \Phi_r(G) =$ positive definite functions on G with $\phi(e) = r$ and $\mathscr{B} = \Phi_{i.d.,s}(G) =$ infinitely divisible positive definite functions on G with $\phi(e) = s$.

(ii) $\mathfrak{A} = \mathscr{B} = \widetilde{\Phi}_1(G) = \operatorname{Log} \Phi_{i.d.,1}(G)$.

The determination of the class of functions that operate in $(\mathfrak{A}, \mathscr{B})$ for other special families may be found in refernces [3]-[8]. Our goal here is to extend the results of [5, 6] and, at the same time, to obtain a new derivation of the results recently announced in [3].

G will denote a locally compact Abelian group and $B^+(G)$ the family of continuous, complex-valued, nonnegative-definite functions on G. Let

$$\begin{aligned}
\Phi_r(G) &= \{\phi : \phi \in B^+(G) \text{ and } \phi(e) = r\}^1 \\
\Phi_{i.d.,r}(G) &= \{\phi : \phi \in \Phi_r(G) \text{ and } (\phi)^{1/n} \in B^+(G) \text{ for } n \geqq 1\} \\
\widetilde{\Phi}_r(G) &= \operatorname{Log} \Phi_{i.d.,r}(G) = \{\log \phi : \phi \in \Phi_{i.d.,r}(G)\} .
\end{aligned}$$

In the case where G is the real line $\Phi_1(G)$ is the class of characteristic functions, $\Phi_{i.d.,1}(G)$ the class of characteristic functions corresponding to the infinitely devisible distributions while $\widetilde{\Phi}_1(G)$ is the class of logarithms of this latter class whose form is well known since Levy and Khintchine.

THEOREM 1. *If G has elements of arbitrarily high order then F operates on $(\Phi_r(G), \Phi_{i.d.,s}(G))$ if and only if*

$$F(z) = s \exp c(f(z/r) - 1) \qquad (|z| \leqq r)$$

where $c \geqq 0$ and

$$f(z) = \sum_{n,m=0}^{\infty} a_{n,m} z^n z^m \qquad (|z| \leqq 1)$$

with

[1] We denote the identity element of G by e.