PACIFIC JOURNAL OF MATHEMATICS
Vol. 25, No. 2, 1968

ON THE TETRAHEDRAL GRAPH

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Generalizing the concept of the triangular association scheme, Bose and Laskar introduced the tetrahedral graph the vertices of which are the $\binom{n}{3}$ unordered triplets selected from n symbols with two points adjacent if and only if their corresponding triplets have two symbols in common. If we let $d(x, y)$ denote the distance between two vertices x, y and $\Delta(x, y)$ the number of vertices adjacent to both x and y, then the tetrahedral graph possesses the following 4 properties:

(B0) the number of vertices is $\binom{n}{3}$

(B1) it is connected and regular of degree $3(n-3)$

(B2) if $d(x, y) = 1$ then $\Delta(x, y) = n - 2$

(B3) if $d(x, y) = 2$ then $\Delta(x, y) = 4$.

The question whether these conditions characterize tetrahedral graphs (no loops or parallel edges permitted) was answered in the affirmative by Bose and Laskar for $n > 16$. In the present paper characterizations of tetrahedral graphs are derived by strengthening each one of (B1), (B2), (B3) and these results are utilized to prove the sufficiency of (B0)-(B3) for $n=6$. (For $n < 4$ the problem is void, $n = 4, 5$ are trivial cases.)

All graphs considered in this paper are finite undirected without loops or parallel edges. As is readily seen the line-graph G of the complete graph with n vertices may be defined as a graph whose vertices are the $\binom{n}{2}$ unordered pairs taken from n symbols so that two pairs are adjacent if and only if they have a symbol in common. Letting $d(x, y)$ denote the distance between x and y and $\Delta(x, y)$ the number of vertices that are adjacent to both x and y, then G has the following properties:

(A0) the number of vertices is $\binom{n}{2}$

(A1) G is connected and regular of degree $2(n-2)$.

(A2) $d(x, y) = 1$ implies $\Delta(x, y) = n - 2$

(A3) $d(x, y) = 2$ implies $\Delta(x, y) = 4$.

Conner [2], Shrikhande [7], Hoffman [3, 4] and Li-chien [5, 6] showed that (A0)-(A3) completely characterize linegraphs of complete graphs except for $n = 8$ where 3 nonisomorphic graphs satisfying (A0)-(A3) exist. Bose and Laskar [1] took up the similar problem concerning unordered triplets chosen from n symbols we mentioned above.

For $n > 16$ (B0)-(B3) characterize tetrahedral graphs as was shown by Bose and Laskar in [1].

For $n < 4$ the characterization problem is meaningless.