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ON WITT'S THEOREM FOR UNIMODULAR QUADRATIC FORMS

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In this paper we give an integral generalization of Witt's theorem for quadratic forms. If J and K are sublattices of a unimodular lattice L, we investigate conditions under which an isometry from J to K will extend to an isometry of L.

Let L be a free $\boldsymbol{Z}$-module (that is a lattice) of finite rank and $\Phi: L \times L \rightarrow \boldsymbol{Z}$ a unimodular symmetric bilinear form on L. We denote $\Phi(\alpha, \beta)$ by $\alpha \cdot \beta$, so that $\alpha \cdot \beta = \beta \cdot \alpha$. A bijective linear mapping $\varphi: J \rightarrow K$, where J and K are sublattices of L, is called an *isometry* if $\varphi(\alpha) \cdot \varphi(\beta) = \alpha \cdot \beta$ for $\alpha, \beta \in J$. Witt's theorem concerns the extension of such an isometry to an isometry of L (onto L). The set of isometries of L form the *orthogonal group* $O(L, \boldsymbol{Z})$ of L.

Vectors α and β in L are called *orthogonal* if $\alpha \cdot \beta = 0$; α^2 denotes $\alpha \cdot \alpha$, the *norm* of α. Any nonzero vector $\alpha \in L$ may be written as $\alpha = d\beta$ with $\beta \in L, d \in \boldsymbol{Z}$ maximal. If $d = 1$, α is called *primitive*; d is the *divisor* of α. It is clear that an isometry φ of L must leave invariant the divisors of all vectors; that is, α and $\varphi(\alpha)$ have the same divisor.

A sublattice U of L is called *primitive* if all the vectors of U which are "primitive in U" are also "primitive in L". In particular the basis vectors of U must be primitive (in L). In considering the extension of an isometry $\varphi: J \rightarrow K$ to an isometry of L, it clearly suffices to consider the case where J and K are primitive sublattices.

A primitive vector $\alpha \in L$ is called *characteristic* if $\alpha \cdot \beta \equiv \beta^2 \pmod 2$ for all $\beta \in L$. Again it is clear that an isometry must map a characteristic vector into a characteristic vector.

Let $r(L)$ and $s(L)$ denote the rank and signature of L. Then we shall prove the following.

THEOREM. *Let $\varphi: J \rightarrow K$ be an isometry between the primitive sublattices J and K of L, where*

$$r(L) - |s(L)| \geqq 2(r(J) + 1) . \tag{1}$$

Then φ extends to an isometry of L if and only if:

α a characteristic vector $\Leftrightarrow \varphi(\alpha)$ a characteristic vector (for each α in J).

This result is a generalization of Wall [1]; in fact we shall use