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SOME QUARTIC DIOPHANTINE EQUATIONS

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Elementary methods are used to solve some quartic Diophantine equations, of which $x^2 = dy^4 + m$ is typical, where m is an integer, positive or negative and d is a positive integer with the property that the equation $x^2 - dy^2 = 4$ has at least one solution x, y where both x and y are odd.

The cases $m = \pm 1, \pm 4$ have been treated previously and in these cases the equations have been solved completely. The object here is to try to extend the method to cover all other values of m. In view of the greater generality of the problem, it is not surprising that the theorems obtained are weaker. However, the method does give a complete solution of the problem in many cases.

In the first place, let us consider the admissible values of d. Clearly if $x^2 - dy^2 = 4$ is to have a solution with x and y both odd, it is necessary that $d \equiv 5 \pmod 8$. Unfortunately this condition is not sufficient, there are five values $d = 37, 101, 141, 189$ and 197 less than 200 which satisfy it without the equation having any odd solutions. There is no simple known necessary and sufficient condition, although several sufficient conditions are known, and these do guarantee the existence of infinitely many such d.

Secondly suppose that $x^2 - dy^2 = 4$ does have a solution in which both x and y are odd. What can be said about the equation $x^2 - dy^2 = -4$? It is easily shown [see 2; § 2] that this will possess a pair of solutions x, y both of which are odd if $x^2 - dy^2 = -1$ has *any* solutions, and none at all otherwise. Again no simple necessary and sufficient conditions are known for the existence of solutions of $x^2 - dy^2 = -1$; it is clearly necessary that d have no factor $\equiv 3 \pmod 4$, and it is known to be sufficient that d be a prime $\equiv 1 \pmod 4$. The main part of this paper will be divided into two parts; in the first we suppose that $x^2 - dy^2 = -4$ has at least one pair of solutions both of which are odd; in the second we suppose that $x^2 - dy^2 = -4$ has no solutions but that $x^2 - dy^2 = 4$ has at least one such set of solutions.

1. We suppose here that $x^2 - dy^2 = -4$ has a pair of solutions x, y both of which are odd. Such values of d have been considered in [1], and we shall use the notation and results from [1]. Thus if the fundamental solution of $x^2 - dy^2 = -4$ is $2\alpha = a + b\sqrt{d}$, then a and b are both odd and the fundamental solutions of the equations $x^2 - dy^2 = 4$, $x^2 - dy^2 = -1$ and $x^2 - dy^2 = 1$ are respectively $2\alpha^2$, α^3