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THE INTEGRATION OF A LIE ALGEBRA REPRESENTATION

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Let $u: G \to A$ be a differentiable representation of a Lie group into a b-algebra. The differential $u_0 = du_e$ of u at the neutral element e of G is a representation of the Lie algebra $\mathfrak{g}$ of G into A. Because a Lie group is locally the union of one-parameter subgroups and since the infinitesimal generator of a differentiable (multiplicative) sub-semi-group of A determines this sub-semi-group, the representation u_0 determines u if G is connected.

We shall be concerned with the converse: given a representation u_0 of $\mathfrak{g}$, when can it be obtained by differentiating a representation u of G? We shall assume G connected and simply connected, which means that we are only interested in the local aspect of the problem.

Call $a \in A$ *integrable* if a differentiable $r: \boldsymbol{R} \to A$ can be found such that $r(s + t) = r(s)r(t)$ and $r'(0) = a$. We can only hope to integrate $u_0: \mathfrak{g} \to A$ to a differentiable $u: G \to A$ if $u_0 x$ is integrable for all $x \in \mathfrak{g}$. We shall prove the

THEOREM. *The set $\mathfrak{h}$ of all elements $x \in \mathfrak{g}$ such that $u_0 x$ is integrable, is a Lie subalgebra of $\mathfrak{g}$; the representation u_0 can be integrated to a representation $u: G \to A$ of the simply connected group G if and only if $\mathfrak{h} = \mathfrak{g}$.*

This result is "best possible" in the following sense:

PROPOSITION 1. *Given a real Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{h}$, there exists a representation $u_0: \mathfrak{g} \to A$ of $\mathfrak{g}$ in a b-algebra A, so that*

$$\mathfrak{h} = \{x \in \mathfrak{g} \mid u_0 x \textit{ is integrable}\} .$$

As a consequence of the theorem, we have the following result: Let x, y be two integrable elements of a b-algebra, and assume that the Lie algebra $\mathfrak{g}$ they generate is finite-dimensional. Then all elements of $\mathfrak{g}$ are integrable.

We cannot drop the assumption that $\mathfrak{g}$ is finite-dimensional. There exists a b-algebra which contains integrable elements x, y such that neither $x + y$ nor $xy - yx$ is integrable.

Elementary properties of b-spaces and b-algebras can be found in [2] or [3]. Differentiable mappings into such spaces are investigated