PACIFIC JOURNAL OF MATHEMATICS
Vol. 26, No. 3, 1968

MULTIPLICITY TYPE AND SUBALGEBRA STRUCTURE IN UNIVERSAL ALGEBRAS

MATTHEW GOULD

By a universal algebra, or briefly, an algebra we shall here mean a pair $\langle A; F\rangle$ consisting of a nonvoid set A and a nonvoid set F of finitary operations on A. The multiplicity type of $\langle A; F\rangle$ is the sequence $\mu = \langle \mu_0, \mu_1, \cdots, \mu_n, \cdots\rangle$ where μ_n is the cardinality of $\{f \in F \mid f \text{ is } n\text{-ary}\}$. The class of all algebras of multiplicity type μ is denoted $K(\mu)$.

We shall study the relationship between the multiplicity type of an algebra and its family of subalgebras. To this end, we set $S(A; F) = \{B \mid \phi \neq B \subseteqq A$ and $\langle B; F\rangle$ is a subalgebra of $\langle A; F\rangle\}$ and, for every multiplicity type μ, $T(\mu) = \{S(A; F) \mid \langle A; F\rangle \in K(\mu)\}$. We define a quasi-ordering $\leqq$ and an equivalence $\equiv$ on the class of multiplicity types as follows. If μ and μ' are multiplicity types, define $\mu \leqq \mu'$ if $T(\mu) \subseteqq T(\mu')$ and $\mu \equiv \mu'$ if $T(\mu) = T(\mu')$. We shall give necessary and sufficient conditions for $\mu \leqq \mu'$, in terms of properties of cardinal numbers, and we shall also find a "normal form" for multiplicity types, whereby every multiplicity type will have a unique representation in normal form and the ordering of multiplicity types in normal form will be characterized by relatively simple criteria.

Our major results, those which characterize the ordering and establish normal form, are Theorems 2.1, 2.2, 2.3, and 2.4.

A family $\mathfrak{A}$ of subsets of a set A is called a *restricted closure system* if whenever $\mathfrak{B} \subseteqq \mathfrak{A}$ and $\bigcap (X \mid X \in \mathfrak{B})$ is nonvoid, then $\bigcap (X \mid X \in \mathfrak{B}) \in \mathfrak{A}$. If $B \subseteqq A$ then the *closure* of B, denoted $[B]$, is defined to be $\bigcap (X \mid X \in \mathfrak{A}, X \supseteqq B)$, provided this intersection is not void. For any algebra $\langle A; F\rangle$ it is easily seen that $S(A; F)$ is a restricted closure system. Birkhoff and Frink [1] proved that for any family $\mathfrak{A}$ of subsets of a nonvoid set A, there is an algebra $\langle A; F\rangle$ such that $\mathfrak{A} = S(A; F)$, if and only if $\mathfrak{A}$ is an *algebraic closure system*, that is, a restricted closure system which is closed under directed union. We shall give a similar result (Theorem 1.1) with a restriction on the multiplicity type.

One minor result of particular interest is the fact that the subalgebra family of any algebra whose operations are finite in number can be realized as the subalgebra family of an algebra having precisely one operation. This is a consequence of Lemma 2.1.

A word on notation: μ and μ' will always denote multiplicity types, and the cardinality of a set A will be denoted $|A|$.