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REGULAR AND IRREGULAR MEASURES ON GROUPS AND DYADIC SPACES

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It is generally known that if X is a σ-compact metric space, then every Borel measure on X is regular. It is not difficult to prove a slightly stronger result, namely that the same conclusion holds if X is a Hausdorff space in which every open subset is σ-compact (I.6 below). The converse is not generally true, even for compact Hausdorff spaces; a counter-example appears here under IV. 1. However, it will be shown in § II that every nondegenerate Borel measure on a nondiscrete locally compact group is regular if and only if the group is σ-compact and metrizable. A similar theorem, proved in § III, holds for dyadic spaces: every Borel measure on such a space is regular if and only if the space is metric.

The result for groups depends on two structure theorems which are proved here: every nonmetrizable compact connected group contains a nonmetrizable connected Abelian subgroup (II.10), and every nonmetrizable locally compact group contains a nonmetrizable compact totally disconnected subgroup (II.11).

In § III, it seems that the separable case requires special attention: a theorem is proved which has as a corollary that every separable dyadic space is a continuous image of $\{0, 1\}^c$ (III.3 and III.4), and one lemma (III.6) uses a weakened version of the continuum hypothesis.

I. Regular and irregular measures.

1. Let X be a topological space, $\mathbf{M}$ a σ-algebra of subsets of X, and μ a (countably additive, nonnegative) measure function whose domain is $\mathbf{M}$. The system $(X, \mathbf{M}, \mu)$ is called *regular measure space* and μ is called a *regular measure* in case

(1) $\mu C < \infty$ for all compact $C \in \mathbf{M}$;

(2) $\mu S = \inf\{\mu U\colon U \text{ open}, U \in \mathbf{M}, U \supset S\}$ for all $S \in \mathbf{M}$;

(3) $\mu U = \sup\{\mu C\colon C \text{ compact}, C \in \mathbf{M}, C \subset U\}$ for all open $U \in \mathbf{M}$.

For lack of a better term, a measure μ will be called *totally regular* if it satisfies the more exclusive definition of regularity favored by some authors (e.g., Halmos in [5]), namely:

$$\mu S = \sup\{\mu C\colon C \text{ compact}, C \in \mathbf{M}, C \subset S\}$$
$$= \inf\{\mu U\colon U \text{ open}, U \in \mathbf{M}, U \supset S\} \text{ for all } S \in \mathbf{M}\,.$$

REMARK 2. According to [7], (10.30) and (10.31), any σ-finite regular measure on a Hausdorff space is totally regular; the proof as