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ON UNICITY OF CAPACITY FUNCTIONS

AKIO OSADA

Sario's capacity function of a closed subset γ of the ideal boundary is known to be unique if γ is of positive capacity. The present paper will determine the number of capacity functions of γ in terms of the Heins harmonic dimension when γ has zero capacity, under the assumption that γ is isolated. This includes the special case where γ is the ideal boundary.

1. Capacity functions. Denote by β the ideal boundary of an open Riemann surface R in the sense of Kerékjártó-Stoïlow. We consider a fixed nonempty closed subset $\gamma \subset \beta$ which is *isolated* from $\delta = \beta - \gamma$. Throughout this paper D will denote a fixed parametric disk about a fixed point $\zeta \in R$ with a fixed local parameter z and the uniqueness is always referred to this fixed triple (ζ, D, z). Here we do not exclude the case where $\gamma = \beta$.

For a regular region $\Omega \supset \bar{D}$ we denote by γ_Ω the part of $\partial\Omega$ which is "homologous" to γ. The remainder $\delta_\Omega = \partial\Omega - \gamma_\Omega$ consists of a finite number of analytic Jordan curves $\delta_{\Omega j}$. For a regular exhaustion $\{R_n\}_{n=0}^\infty$ with $R_0 \supset \bar{D}$ and nonempty γ_{R_0}, set $\gamma_n = \gamma_{R_n}$ and $\delta_{nj} = \delta_{R_n j}$. Then there exists a unique function $p_{\gamma_n} \in H(R_n - \zeta)$ satisfying

(a) $p_{\gamma_n} \mid D = \log|z - \zeta| + h_n(z)$ with $h_n \in H(\bar{D})$ and $h_n(\zeta) = 0$,

(b) $p_{\gamma_n} \mid \gamma_n = k_n(\gamma)$ (const.) and $p_{\gamma_n} \mid \delta_{nj} = d_{nj}$ (const.) so that $\int_{\delta_{nj}} * dp_{\gamma_n} = 0$, which is called a capacity function of γ_n (Sario [6]). It is known that $k_n(\gamma)$ increases with n and the limit $k(\gamma)$ is independent of the choice of $\{R_n\}_{n=0}^\infty$. We call $e^{-k(\gamma)}$ the capacity of γ and denote it by cap γ. When cap $\gamma > 0$, p_{γ_n} converges to a functions p_γ, which is independent of the choice of the exhaustion (Sario [6]). Even when cap $\gamma = 0$, we can also choose a subsequence of $\{p_{\gamma_n}\}$ which converges to a function p_γ. Such functions p_γ will be called capacity functions of γ (Sario [6]). As mentioned above there exists only one capacity function when cap $\gamma > 0$.

It is the purpose of this paper to determine the number of capacity functions p_γ when cap $\gamma = 0$.

2. The harmonic dimension of γ. Let R, β, γ and δ be as in 1. Furthermore we suppose that γ *is of zero capacity*. For a regular region $\Omega \supset \bar{D}$ we denote by $V_{\Omega i}$ components of $R - \bar{\Omega}$ whose derivations are contained in γ and by $W_{\Omega j}$ the remaining components. Here an ideal boundary component will be called a derivation of $V_{\Omega i}$ when it is contained in the closure of $V_{\Omega i}$ in the compactification of R. Here-