PACIFIC JOURNAL OF MATHEMATICS
Vol. 28, No. 2, 1969

LIOUVILLE'S THEOREM

ROBERT PHILLIPS

Liouville's theorem states that in Euclidean space of dimension greater than two, every conformal mapping must, by necessity, be an elementary transformation (i.e., a translation, a magnification, an orthogonal transformation, a reflection through reciprocal radii, or a combination of these transformations). This theorem was proven by R. Nevanlinna under the additional assumption that the mappings be at least four times differentiable. We show that a modified version of Nevanlinna's proof is still valid when the mappings are assumed to be only twice differentiable. Our methods are those of Nonstandard Analysis as developed by A. Robinson.

We begin with a brief introduction to this subject.

1. Nonstandard Analysis.

It is known that there exist proper extensions of the real numbers which possess the same formal properties as the real numbers [Robinson [1]). That is, given a formal mathematical language L in which the algebraic and topological properties of the real numbers R can be expressed, there will exist a proper set-theoretical extension $^*R$ of R with the following property: any sentence of L which holds (in the model-theoretic sense) in R will hold in $^*R$. Then, since we have assumed that the ordered field axioms which hold in R are expressible in L, $^*R$ will be an ordered field; whence, it follows that $^*R$ is nonarchimedian. Therefore, there will exist in $^*R$ elements $a \neq 0$ that have the property that $|a| < r$ for all positive r in R. These elements are called infinitesimal. If $a - b$ is an infinitesimal, we shall write $a \approx$ b. If "0" is regarded as an infinitesimal then it is clear that "$\approx$" is an additive congruence relation on $^*R$. Because R has the formal property "that for each positive number r there will exist a natural number n so that $n > r$", $^*R$ also has this property. Thus since $^*R$ contains infinite numbers (reciprocals of infinitesimals), it follows that $^*R$ has infinite natural numbers. That is, embedded in $^*R$ is a proper set-theoretical extension of $^*N$ of N (N denotes the natural numbers) which has the same formal properties as N. The numbers in $^*N - N$ are just the infinite natural numbers.

If E_n denotes Euclidean space of dimension n, we denote by $^*E_n$, the natural extension of E_n induced by $^*R$. That is, elements of $^*E_n$ are ordered n-tuples of elements of $^*R$. If $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$ are elements of $^*E_n$ then $a \approx b$ means