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TORSION IN BBSO

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The cohomology of BBSO, the classifying space for the stable Grassmanian BSO, is shown to have torsion of order precisely 2^r for each natural number r. Moreover, the elements of order 2^r appear in a pattern of striking simplicity.

Many of the stable Lie groups and homogeneous spaces have torsion at most of order 2 [1, 3, 5]. There is one such space, however, with interesting torsion of higher order. This is $BBSO = SU/\mathrm{Spin}$ which is of interest in connection with Bott periodicity and in connection with the J-homomorphism [4, 7]. By the notation SU/Spin we mean that $BBSO$ can be regarded as the fibre of $B\ \mathrm{Spin} \to BSU$ or that, up to homotopy, there is a fibration

$$SU \to BBSO \to B\,\mathrm{Spin}$$

induced from the universal SU bundle by $B\ \mathrm{Spin} \to BSU$. The mod 2 cohomology $H^*(BBSO; Z_2)$ has been computed by Clough [4]. The purpose of this paper is to compute enough of $H^*(BBSO; Z)$ to obtain the mod 2 Bockstein spectral sequence [2] of $BBSO$.

Given a ring R, we shall denote by $R[x_i \mid i \in I]$ the polynomial ring on generators x_i indexed by elements of a set I. The set I will often be described by an equation or inequality in which case i is to be understood to be a natural number. Similarly $E(x_i \mid i \in I)$ will denote the exterior algebra on generators x_i. In this case, we will need only $R = Z_2$.

Let us recall the results on B Spin as given by Thomas [6] and on $BBSO$ as given by Clough [4].

$$H^*(B\,\mathrm{Spin}; Z_2) \approx Z_2[w_i \mid i \neq 2^j + 1]$$

where w_i is (the image of) the Stiefel-Whitney class w_i.

$$H^*(B\,\mathrm{Spin}; Z) \approx Z[Q_i \mid i > 0] \oplus \hat{T}$$

where $2\hat{T} = 0$ and $Q_i \in H^{4i}$.

$$H^*(BBSO; Z_2) \approx E(e_i \mid i \geqq 3)$$

where $e_i \in H^i$ and is the image of w_i if $i \neq 2^j + 1$ while e_{2^j+1} maps to an indecomposable element in $H^*(SU; Z_2)$.

Now let ${}_\beta E_r$ denote the mod 2 Bockstein spectral sequence of $BBSO$ [2]. In particular, ${}_\beta E_2 = \mathrm{Ker}\, Sq^1/\mathrm{Im}\, Sq^1$. Now $Sq^1 w_{2i} = w_{2i+1}$ in BSO and $Sq^1 w_{2i+1} = 0$ while $Sq^1 e_{2^j} = 0$ in B Spin. We will see that