PACIFIC JOURNAL OF MATHEMATICS
Vol. 28, No. 3, 1969

A NOTE ON RECURSIVELY DEFINED ORTHOGONAL POLYNOMIALS

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Let $\{a_i\}_{i=0}^{\infty}$ and $\{b_i\}_{i=0}^{\infty}$ be real sequences and suppose the b_i,s are all positive. Define a sequence of polynomials $\{P_i(x)\}_{i=0}^{\infty}$ as follows: $P_0(x) = 1$, $P_1(x) = (x - a_0)/b_0$, and for $n \geqq 1$

$$(*) \qquad b_n P_{n+1}(x) = (x - a_n)P_n(x) - b_{n-1}P_{n-1}(x) .$$

Favard showed that the polynomials $\{P_i(x)\}$ are orthonormal with respect to a bounded increasing function ψ defined on $(-\infty, +\infty)$. This note generalizes recent constructive results which deal with connections between the two sequences $\{a_i\}$ and $\{b_i\}$ and the spectrum of ψ. (The spectrum of ψ is the set $S(\psi) = \{\lambda : \psi(\lambda + \varepsilon) - \psi(\lambda - \varepsilon) > 0$ for all $\varepsilon > 0\}$.) It is shown that if $b_i \to 0$ then every limit point of the sequence $\{a_i\}$ is in $S(\psi)$.

2. Preliminaries. In order to use theorems from functional analysis, consider the space $\mathscr{L}^2(\psi) = \{f : \int_{-\infty}^{+\infty} f^2 d\psi < \infty\}$. This is a Hilbert space where the inner product is gived by $(f, g) = \int f g \, d\psi$ and where we identify all functions which agree on $S(\psi)$. In [2], (p. 215), Carleman showed that the condition $\sum 1/\sqrt{b_i} = \infty$ implies that when ψ is normalized to be continuous from the left and to have $\psi(-\infty) = 0$, $\psi(+) = 1$, then it is unique. In [6], M. Riesz showed that if ψ is essentially unique then Parseval's relation holds for the orthonormal set $\{P_i\}$ in the space $\mathscr{L}^2(\psi)$. Hence the set $\{P_i\}$ is dense in this space.

We now make the assumption that $\lim b_i = 0$. Combining the Carleman result and the Riesz result we see that ψ is essentially unique and the polynomials $\{P_i\}$ are a dense set in $\mathscr{L}^2(\psi)$. Using this information we define an operator A on a dense subset of $\mathscr{L}^2(\psi)$. The domain of A is the set of all functions f which are in $\mathscr{L}^2(\psi)$ and for which xf is also in $\mathscr{L}^2(\psi)$. We take A to be the self-adjoint operator defined by $(Af)(x) = xf(x)$. By inspection of $(*)$ we see that for $i = 1, 2, 3, \cdots$ we have

$$(**) \qquad A(P_i) = b_{i-1}P_{i-1} + a_i P_i + b_i P_{i+1} .$$

We call A the operator associated with the sequences $\{a_i\}$ and $\{b_i\}$.

3. Theorems. Let $\sigma(A)$ be the spectrum of the operator A, i.e., all points λ where $A - \lambda I$ does not have a bounded inverse. Then we have the following: