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INITIAL SEGMENTS OF ONE-ONE DEGREES

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Let 0 be the one-one degree consisting of the infinite recursive sets whose complements are also infinite. In this paper are studied initial segments of the one-one degrees $\geqq \mathbf{0}$. A characterization is stated of the order types of those initial segments with greatest member which are at the same time initial segments of the many-one degrees. It is shown that any finite initial segment with greatest member is a lattice, and that any finite recursively enumerable initial segment with greatest member is a distributive lattice.

According to the usual definition of one-one reducibility a set of natural numbers A is one-one reducible to a set of natural numbers B written $A \leqq_1 B$ if there is a one-one recursive function ρ such that $\rho^{-1}(B) = A$. The relation of one-one equivalence $\equiv_1$ defined by

$$A \equiv_1 B \Leftrightarrow [A \leqq_1 B \quad \text{and} \quad B \leqq_1 A]$$

is an equivalence relation, and the equivalence classes into which it partitions the sets of natural numbers are called one-one degrees. The natural ordering of these degrees induced by $\leqq_1$ is denoted by $\leqq$. In this paper we shall be wholly concerned with the one-one degrees $\geqq \mathbf{0}$. Notice that if A has one-one degree $\geqq \mathbf{0}$ then neither A nor its complement is immune. In this context it is convenient to adopt a new definition of one-one reducibility: we say that A is *one-one reducible* to a set of natural numbers B, again written $A \leqq_1 B$, if there exists a one-one partial recursive (p.r.) function ρ such that $\operatorname{dom} \rho$ (the domain of ρ) and $A \cup \operatorname{dom} \rho$ are both recursive, and such that $\rho^{-1}(B) = A \cap \operatorname{dom} \rho$. On sets whose one-one degrees in the old sense are $\geqq \mathbf{0}$ the new definition of $\leqq_1$ is the same as the old. Also, for any set A there is a set B whose one-one degree in the old sense is $\geqq \mathbf{0}$, and such that $A \equiv_1 B$ in the new sense. Thus adopting this new definition leaves the one-one degrees $\geqq \mathbf{0}$ unchanged, and suppresses the remaining one-one degrees.

We adopt the usual definition of many-one reducibility save that $\varnothing$, N (the set of all natural numbers) are both by convention in the zero many-one degree consisting of all recursive sets.

The only notations likely to be unfamiliar consist in writing $A \oplus B$ for the set $\{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$, A' for the complement of A, and $\operatorname{rng} \rho$ for the range of ρ.

1. Common initial segments of the many-one and one-one degrees. In [3] we obtained a characterization of the isomorphism