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LOCAL DOMAINS WITH TOPOLOGICALLY T-NILPOTENT RADICAL

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This paper is concerned with local integral domains (no chain condition) which have the following property: for each ideal $\mathscr{A} \neq 0$ of A and for each sequence $(a_n)_{n\in N}$ of elements of $M(A)$, the maximal ideal of A, there is an $M \in N$ such that $a_0 a_1 \cdots a_k \in \mathscr{A}$. A local domain with this property is called a local domain with TTN. These rings are shown to be rings with Krull dimension 1 and local domains with Krull dimension 1 are shown to be dominated by rank 1 valuation rings. Modules over these rings are studied and results concerning divisibility and existence of simple submodules are obtained.

Noetherian integral domains with TTN are studied. Integral extensions of these rings are also studied. By localization of previous results, a characterization is given of those integral domains A with the property that every nonzero torsion A-module has a simple submodule.

H. Bass in [1] studied rings with the property that the Jacobson radical was T-nilpotent (T for transfinite), i.e., for each sequence $(a_n)_{n\in N}$ of elements of the Jacobson radical, $a_0 a_1 \cdots a_k = 0$ for some k. Local integral domains with TTN are just local domains with the property that $A/\mathscr{A}$ has T-nilpotent radical for each ideal $\mathscr{A} \neq 0$ of A.

In this paper A will denote a ring. All rings will be assumed to be commutative and have an identity. All modules will be unitary.

A will be called a local ring if A has a unique maximal ideal. If A is a local ring, $M(A)$ will denote its maximal ideal. If B is a local subring of A, A is said to dominate B if $M(A) \cap B = M(B)$. For convenience we agree that an integral domain is not a field.

If E and F are A-modules, $E \otimes F$ will mean $E \otimes_A F$.

DEFINITION. An ideal $\mathscr{A}$ of a ring A will be called topologically T-nilpotent if for each ideal $\mathscr{B}$ of A, $\mathscr{B} \subset \mathscr{A}$, $\mathscr{B} \neq 0$, and each sequence $(a_i)_{i\in N}$ of elements of $\mathscr{A}$, there is an $n \in N$ with $a_0 a_1 \cdots a_n \in \mathscr{B}$.

DEFINITION. An ideal $\mathscr{A}$ of a ring A will be called topologically nilpotent if for each ideal $\mathscr{B}$ of A, $\mathscr{B} \subset \mathscr{A}$, $\mathscr{B} \neq 0$, and each element a of $\mathscr{A}$, $a^n \in \mathscr{B}$ for some $n \in N$.

It is clear that it suffices to consider ideals $\mathscr{B}$ which are nonzero principal ideals.