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THE PART METRIC IN CONVEX SETS

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Any convex set C without lines in a linear space L can be decomposed into disjoint convex subsets (called parts) in a way which generalizes the idea of Gleason parts for a function space or function algebra. A metric d (called part metric) can be defined on C in a purely geometric way such that the parts of C are the components in the d-topology. This paper treats the connection between the convex structure of C and the metric d. The situation is particularly interesting when C is closed with respect to a weak Hausdorff topology on L (defined by a duality between L and another linear space). Then C is characterized by the set C^+ of all continuous affine functions F on L satisfying $F(x) \geqq 0$ for all $x \in C$. This allows us to define d in terms of the functions $\log F$, $F \in C^+$. Furthermore, d-completeness of C can be derived from the completeness of C in L. The "convexity" of the metric d leads to the existence of a continuous selection function for lower semi-continuous mappings of a paracompact space into the nonempty d-closed convex subsets of one part of such a complete convex set C. We apply this result and the study of the part metric of the convex cone of positive Radon measures on a locally compact Hausdorff space to the problem of selecting in a continuous way mutually absolutely continuous representing measures for points in one part of a function space or function algebra.

1. The part metric and convex structure. We consider a real linear space L, and a convex set C in L which contains no whole line. We do not necessarily assume that L has a topology.

The closed segment from x to y is denoted $[x, y]$. If $x, y \in C$, we say that $[x, y]$ extends (in C) by $r(>0)$ if $x + r(x - y) \in C$ and $y + r(y - x) \in C$. We write $x \sim y$ if $[x, y]$ extends by some $r > 0$. It is shown in [1] that $\sim$ defines an equivalence relation in C.

The equivalence classes of $\sim$, called the *parts* of C, are clearly also convex. There is a metric d on each part of C defined by

$$d(x, y) = \inf \left\{\log\left(1 + \frac{1}{r}\right): [x, y] \text{ extends by } r\right\} .$$

If $[x, y]$ extends by r (in C), then $x + r'(x - y)$ and $y + r'(y - x)$ are in the part Π of x and y for all $r' < r$. It follows that one gets the same part metric on Π if one replaces C by Π in the definition of $d(x, y)$.

If $x \not\sim y$, we write $d(x, y) = +\infty$. Then d satisfies all axioms of