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# MULTI-VALUED CONTRACTION MAPPINGS

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**Some fixed point theorems for multi-valued contraction mappings are proved, as well as a theorem on the behaviour of fixed points as the mappings vary.**

In § 1 of this paper the notion of a multi-valued Lipschitz mapping is defined and, in § 2, some elementary results and examples are given. In § 3 the two fixed point theorems for multi-valued contraction mappings are proved. The first, a generalization of the contraction mapping principle of Banach, states that a multi-valued contraction mapping of a complete metric space $X$ into the nonempty closed and bounded subsets of $X$ has a fixed point. The second, a generalization of a result of Edelstein, is a fixed point theorem for compact set-valued local contractions. A counterexample to a theorem about $(\varepsilon, \lambda)$-uniformly locally expansive (single-valued) mappings is given and several fixed point theorems concerning such mappings are proved.

In § 4 the convergence of a sequence of fixed points of a convergent sequence of multi-valued contraction mappings is investigated. The results obtained extend theorems on the stability of fixed points of single-valued mappings [19].

The classical contraction mapping principle of Banach states that if $(X, d)$ is a complete metric space and $f: X \rightarrow X$ is a contraction mapping (i.e., $d(f(x), f(y)) \leqq \alpha d(x, y)$ for all $x, y \in X$, where $0 \leqq \alpha < 1$), then $f$ has a unique fixed point. Edelstein generalized this result to mappings satisfying a less restrictive Lipschitz inequality such as local contractions [4] and contractive mappings [5]. Knill [13] and others have considered contraction mappings in the more general setting of uniform spaces.

Much work has been done on fixed points of multi-valued functions. In 1941, Kakutani [10] extended Brouwer's fixed point theorem for the $n$-cell to upper semi-continuous compact, nonempty, convex set-valued mappings of the $n$-cell. In 1946 Eilenberg and Montgomery [7] generalized Kakutani's result to acyclic absolute neighborhood retracts and upper semicontinuous mappings $F$ such that $F(x)$ is nonempty, compact, and acyclic for each $x$. In 1953, Strother [22] showed that every continuous multi-valued mapping of the unit interval of $I$ into the nonempty compact subsets of $I$ has a fixed point but that the analogous result for the 2-cell is false. In [22] Strother also proved some fixed point theorems for multi-valued mappings with restrictions on the manner in which the images of points are embedded under a homeomorphism of the space onto a retract of a Tychonoff