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A RADICAL COINCIDING WITH THE LOWER RADICAL IN ASSOCIATIVE AND ALTERNATIVE RINGS

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In a recent paper by the second author a construction was given which was shown to coincide with the lower radical in all associative rings. In the present paper this construction is considered in various classes of not necessarily associative rings. It is shown that while the construction still defines a radical, it will in general properly contain the lower radical. More precisely, it is shown that the radical constructed coincides with the lower radical if the semisimple class of the lower radical is hereditary (or, equivalently, if the radical of a ring always contains the radicals of all its ideals).

From this condition it follows that the construction coincides with the lower radical in all associative and alternative rings, but an example is given which shows that this is not true in general. We conclude by showing that an apparently quite different construction due to J. F. Watters [5] yields exactly the same class of rings.

We will assume that all rings considered in this paper are from some universal class $\mathscr{U}$ of not necessarily associative rings. We will use the following construction, which is equivalent to that of [4]. Let $\mathscr{A}$ be an arbitrary class of rings and $\mathscr{A}_0$ its homomorphic closure. Then define $\mathscr{A}_n = \{R \in \mathscr{U} \mid R$ has a nonzero ideal $I \in \mathscr{A}_{n-1}\}$, and $A_\omega = \bigcup_n \mathscr{A}_n$. Then define $\mathscr{Y}(\mathscr{A}) = \{R \in \mathscr{U} \mid R/I \in \mathscr{A}_\omega$ for all ideals I of $R\}$. It is clear from this definition that we have

LEMMA 1. $\mathscr{A} \subseteqq \mathscr{A}_0 \subseteqq \mathscr{Y}(\mathscr{A})$.

LEMMA 2. $\mathscr{A} \subseteqq \mathscr{B}$ *implies* $\mathscr{Y}(\mathscr{A}) \subseteqq \mathscr{Y}(\mathscr{B})$.

It is also easy to check that the proof of [4, Th. 1] makes no use of associativity. Thus we may state

THEOREM 1. $\mathscr{Y}(\mathscr{A})$ *is a radical class.*

We will replace [4, Th. 2] by the following generalization:

THEOREM 2. *If* $\mathscr{P}$ *is a radical sub-class of* $\mathscr{U}$, *then* $\mathscr{P} = \mathscr{Y}(\mathscr{P})$ *if either of the following two equivalent conditions is satisfied*:
(i) *The semisimple class* $\mathscr{S}\mathscr{P}$ *of* $\mathscr{P}$ *is hereditary*,
(ii) *Writing* $\mathscr{P}(R)$ *for the* $\mathscr{P}$*-radical of* R, *then* $\mathscr{P}(I) \subseteqq \mathscr{P}(R)$