PACIFIC JOURNAL OF MATHEMATICS
Vol. 30, No. 3, 1969

POLYHEDRON INEQUALITY AND STRICT CONVEXITY

B. B. PHADKE

This paper considers convexity of functions defined on the "Grassmann cone" of simple r-vectors. It is proved that the strict polyhedron inequality does not imply strict convexity.

H. Busemann, in conjunction with others, (see [3]), has considered the problem of giving a suitable definition of the convexity of functions defined on nonconvex sets. An examination of various methods of defining convexity on the "Grassmann cone" (see [1]) is found in [2]. The most important open problems (see [3]) are whether weak convexity implies the area minimizing property (also called the polyhedron inequality) and whether the latter implies convexity. A modest result in this direction is proved below, namely, the strict area minimizing property does not imply strict convexity.

2. **Basic definitions.** Let a continuous function $\mathscr{F}$ be defined on the Grassmann cone G_r^n of the simple r-vectors R in the linear space V_r^n of all r-vectors $\tilde{R}$ (over the reals). Let $\mathscr{F}$ be positive homogeneous, i.e., $\mathscr{F}(\lambda R) = \lambda \mathscr{F}(R)$ for $\lambda \geqq 0$. To a Borel set F in an oriented r-flat $\mathscr{R}^+$ in the n-dimensional affine space A^n, we associate a simple r-vector as follows: $R = 0$ if F has r-dimensional measure 0, and otherwise $R = v_1 \wedge v_2 \wedge \cdots \wedge v_r$, is parallel to $\mathscr{R}^+$ and the measure of the parallelepiped spanned by $v_1, v_2, \cdots, v_r$ equals the measure of F. (Note a set of measure 0 and equality of measures in parallel r-flats are affine concepts and hence welldefined.) We denote below by $\mathscr{R}$ an r-flat parallel to an r-vector R passing through the origin.

DEFINITION 1. We say that $\mathscr{F}$ has the strict area minimizing property ($SFMA$) if: Whenever $R_0, R_1, \cdots, R_p$ are associated to r-dimensional faces of an r-dimensional oriented closed polyhedron P we have $\mathscr{F}(-R_0) < \Sigma \mathscr{F}(R_i)$, with $i = 1$ to p, unless $R_i = \lambda_i R_0$, $\lambda_i \geqq 0$ for all $i = 1$ to p (called the strict Polyhedron Inequality).

DEFINITION 2. $\mathscr{F}$ is said to be strictly weakly convex (SWC) if: Whenever R, R_1 and R_2 are simple, $R = R_1 + R_2, R_1$ is not a scalar multiple of R_2, we have $\mathscr{F}(R) < \mathscr{F}(R_1) + \mathscr{F}(R_2)$.

DEFINITION 3. $\mathscr{F}$ is said to be convex (C) if there exists a convex extension of $\mathscr{F}$ to V_r^n.