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A MONOTONICITY PRINCIPLE FOR EIGENVALUES

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The smallest eigenvalue of certain boundary problems for second order linear elliptic partial differential equations increases to infinity as the domain in question shrinks to the empty set. The object of this note is to formulate and prove an analogous result for linear elliptic differential operators L of general even order. Specifically, let $G(t)$ be a bounded domain in n-dimensional Euclidean space, and suppose that $G(t)$ has thickness t (in a sense which will be precisely defined below). Let $\lambda_0(t)$ be the smallest eigenvalue of a boundary problem associated with L and $G(t)$. It will be shown that $\lambda_0(t)$ increases to infinity as t tends to zero from the right.

The proof depends on a generalization of Agmon's form [1] of Poincaré's inequality. In the second-order case, a monotonicity principle of the type under consideration has been applied to obtain oscillation theorems (cf. [3], [4]) for partial differential equations on unbounded domains.

2. Preliminary lemmas. Let G be a domain (not necessarily bounded) in n-dimensional Euclidean space R^n. We shall say that G has *bounded thickness* $\leqq s$, or simply *thickness* $\leqq s$, if and only if there is a line ℓ such that each line parallel to ℓ intersects G in a set each of whose components (i.e., maximal connected subsets) has diameter $\leqq s$. For example, if $|x|$ denotes the length $(\sum x_i^2)^{1/2}$ of the vector $x = (x_1, \cdots, x_n)$ in R^n, then the annulus $\{x \in R^n : r_0 < |x| < r_1\}$, $r_0 > 0$, has thickness $\leqq 2\sqrt{[r_1^2 - r_0^2]}$.

Let $C^m(G)$ denote the class of all m times continuously differentiable real-valued functions on G, and $C_0^m(G)$ denote the class of all C^m functions having compact support in G. We use the standard multi-index notation: let $\alpha = (\alpha_1 \cdots, \alpha_n)$ have nonnegative integral components and "norm" $|\alpha| = \alpha_1 + \cdots + \alpha_n$; let $D_i^{\alpha_i}$ denote the partial differential operator $(\partial^{\alpha_i}/\partial x_i^{\alpha_i})$, and let $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$.

LEMMA 1. *If G has bounded thickness $\leqq s$ and if every line parallel to the line ℓ in the definition of bounded thickness intersects G in a set with at most k components, where k is some positive integer, then*

$$|v|_{j,G} \leqq (ks)^{m-j} |v|_{m,G} \tag{1}$$

for all $v \in C_0^m(G)$, $0 \leqq j \leqq m-1$, where