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INJECTIVE HULLS OF SEMI-SIMPLE MODULES OVER REGULAR RINGS

A. K. TIWARY

The object of this paper is to provide an explicit construction of the injective hull of a semi-simple module over a commutative regular ring.

The existence of injective hulls of an arbitrary module M and their uniqueness upto isomorphism over M was shown by B. Eckmann and A. Schopf in 1953 [6]. But only in few cases these hulls have been described explicitly [1, 2].

In the special case when the ring is regular as well as Noetherian, the problem is already solved since over such a ring every module is known to be semi-simple [9] and hence is its own injective hull [11, 10]. To begin with we show that every monotypic component of the module is injective and then prove a topological lemma about T_1-spaces. The Zariski topology of the maximal ideal space of the basic ring being T_1, we make use of the lemma to obtain the desired construction of an injective hull of the module. We show by an example that a semi-simple module over a regular ring need not always be injective and obtain finally a necessary and sufficient condition for the injectivity of the module.

DEFINITION 1. A ring R is called (von Neumann) *regular* if for every $a \in R$, there exists an element $x \in R$ such that $axa = a$. This condition reduces to $a^2x = a$ if R is commutative. A Boolean ring is an example of a commutative regular ring. It is well known that a commutative ring R with unit is regular if and only if every simple R-module is injective [11].

Throughout this paper we shall consider R to be a commutative regular ring with unit 1. Let Ω denote the set of maximal ideals of R. For each $a \in R$ define Ω_a by $\Omega_a = \{P \in \Omega \mid a \notin P\}$. It follows that $\Omega_a \cap \Omega_b = \Omega_{ab}$. Thus Ω can be made into a topological space with $\{\Omega_a \mid a \in R\}$ as the system of basic open sets. This topology of Ω is known as the Zariski topology. Ω is clearly a T_1-space since if P and Q are any two distinct points in Ω, there exists $a \in P - Q$ which implies that Ω_a is a neighbourhood of Q not containing P.

DEFINITION 2. Let M be a semi-simple R-module. For any simple submodule S of M, there exists exactly one $P \in \Omega$ with $S \cong R/P$. The