PACIFIC JOURNAL OF MATHEMATICS
Vol. 31, No. 1, 1969

SYMMETRIC POSITIVE DEFINITE MULTILINEAR FUNCTIONALS WITH A GIVEN AUTOMORPHISM

MARVIN MARCUS AND STEPHEN PIERCE

Let V be an n-dimensional vector space over the real numbers R and let φ be a multilinear functional,

$$\varphi\colon \overset{m}{\underset{1}{\times}} V \longrightarrow R \tag{1}$$

i.e., $\varphi(x_1, \cdots, x_m)$ is linear in each x_j separately, $j = 1, \cdots, m$. Let H be a subgroup of the symmetric group S_m. Then φ is said to be *symmetric* with respect to H if

$$\varphi(x_{\sigma(1)}, \cdots, x_{\sigma(m)}) = \varphi(x_1, \cdots, x_m) \tag{2}$$

for all $\sigma \in H$ and all $x_j \in V$, $j = 1, \cdots, m$. (In general, the range of φ may be a subset of some vector space over R.) Let $T\colon V \rightarrow V$ be a linear transformation. Then T is an *automorphism* with respect to φ if

$$\varphi(Tx_1, \cdots, Tx_m) = \varphi(x_1, \cdots, x_m) \tag{3}$$

for all $x_i \in V$, $i = 1, \cdots, m$. It is easy to verify that the set $\mathfrak{A}(H, T)$ of all φ which are symmetric with respect to H and which satisfy (3) constitutes a subspace of the space of all multilinear functionals symmetric with respect to H. We denote this latter set by $M_m(V, H, R)$.

We shall say that φ is *positive definite* if

$$\varphi(x, \cdots, x) > 0 \tag{4}$$

for all nonzero x in V, and we shall denote the set of all positive definite φ in $\mathfrak{A}(H, T)$ by $P(H, T)$. It can be readily verified that $P(H, T)$ is a convex cone in $\mathfrak{A}(H, T)$.

Our main results follow.

THEOREM 1. *If $P(H, T)$ is nonempty then*

(a) *m is even*

and

(b) *every eigenvalue of T has modulus* 1.

If, in addition, T has only real eigenvalues then

(c) *every elementary divisor of T is linear.*

Conversely if (a), (b) *and* (c) *hold then $P(H, T)$ is nonempty. Moreover, if $P(H, T)$ is nonempty then $\mathfrak{A}(H, T)$ is the linear closure of $P(H, T)$.*

In Theorem 2 we shall investigate the dimension of $\mathfrak{A}(H, T)$ in the event $P(H, T)$ is not empty. To do this we must introduce some combinatorial notation. Let $\Gamma_{m,n}$ denote the set of all sequences