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ON THE SUM $\sum \langle n\alpha\rangle^{-t}$ AND NUMERICAL INTEGRATION

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Let "$\langle x\rangle$" denote the distance of the real number x from the nearest integer. If α is an irrational number, the growth of the sum

$$\sum_{K<n\leqq AK} \langle n\alpha\rangle^{-t}$$

(A is a fixed number, >1) as $K\to\infty$ depends on the nature of the rational approximations to α. We shall find estimates of this sum, for certain classes of irrational numbers. Part of the motivation for these estimates is an application to Korobov's theory of numerical evaluation of multiple integrals.

A few years ago N. M. Korobov [8], [9] (and independently E. Hlawka [4]) invented a number-theoretical method for the numerical integration of periodic functions of several variables. Let $E_s^n(C)$, $n>1$, be the set of all functions f of s real variables having period 1 in each variable, and whose Fourier expansion

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{m}} C(\boldsymbol{m})e^{2\pi i\boldsymbol{x}\cdot\boldsymbol{m}} \tag{1}$$

(here $\boldsymbol{x}$ and $\boldsymbol{m}$ are s-tuples, of real numbers and of integers respectively, and the sum is over all possible $\boldsymbol{m}$) satisfies the condition

$$|C(\boldsymbol{m})| \leqq C\Big(\prod_{i=1}^{s}\max(1, |m_i|)\Big)^n . \tag{2}$$

We shall denote the product inside the parentheses in (2) by "$\|\boldsymbol{m}\|$". (It is not a norm in the usual sense.)

Let G_s be the unit cube in s-space. Korobov considered the approximation of

$$I = I(f) = \int_{G_s} f(\boldsymbol{x})d\boldsymbol{x}$$

by the sum

$$Q(f) = Q(f, N, \boldsymbol{a}) = \frac{1}{N}\sum_{r=1}^{N} f(r\boldsymbol{a}) \; ; \tag{3}$$

the problem is to choose $\boldsymbol{a} = \boldsymbol{a}(N) = (a_1(N), \cdots, a_s(N))$ so that $|Q-I|$ will go to zero rapidly as N increases. He made the following definition: ([9], p. 96; we have modified the form slightly).

DEFINITION. Let $N_1, N_2, \cdots$ be an increasing sequence of positive integers. Then a sequence $\boldsymbol{a}(N_1), \boldsymbol{a}(N_2), \cdots$ of s-tuples of integers is