PACIFIC JOURNAL OF MATHEMATICS
Vol. 31, No. 3, 1969

THE CONTENT OF SOME EXTREME SIMPLEXES

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Formulae are presented that give the content of a simplex in Euclidean n-space: (i) in terms of the lengths of and the angles between the vectors from a fixed point to the vertices of the simplex; (ii) in terms of the lengths of and the angles between the perpendiculars from a fixed point to the bounding faces of the simplex. We then determine the largest simplex whose vertices are given distances from a fixed point and we determine the smallest simplex whose faces are given distances from a fixed point. As special cases we find that the regular simplex is the largest simplex contained in a given sphere and is also the smallest simplex containing a given sphere.

1. Introduction and results. The n-dimensional simplex S_n in Euclidean n-space is the general term in the sequence of figures S_0, $S_1, S_2, S_3 \cdots$ known respectively otherwise as point, line segment, triangle, tetrahedron, $\cdots$. S_n is determined by $n+1$ points, P_1, $P_2, \cdots, P_{n+1}$, – its vertices –, which we assume do not lie in any $(n-1)$-dimensional hyperplane. Taken n at a time, these vertices determine $(n-1)$-dimensional hyperplanes $H_1, H_2, \cdots, H_{n+1}$, where H_i contains all vertices except P_i. We choose the normal of H_i so that P_i lies on the negative side of H_i. S_n can be regarded as the intersection of these $n+1$ nonpositive half spaces; it can also be regarded as the convex hull of its vertices.

Let Q be an arbitrary point. For $i = 1, 2, \cdots, n+1$, let $d_i > 0$ be the distance from Q to P_i and let $e_i > 0$ be the distance from Q to H_i. Let $\boldsymbol{a}_i$ be the unit vector in the direction from Q to P_i and let $\boldsymbol{b}_i$ be the unit vector from Q along the perpendicular to H_i. Let $r_{ij} = \boldsymbol{a}_i \cdot \boldsymbol{a}_j$, $s_{ij} = \boldsymbol{b}_i \cdot \boldsymbol{b}_j$, $i, j = 1, 2, \cdots, n+1$.

In this paper, we first show that the content, V_n, of S_n is given by

$$n!\, V_n = \left| \sum_{i,j} R_{ij} \frac{1}{d_i} \frac{1}{d_j} \right|^{1/2} \prod_1^{n+1} d_i \tag{1}$$

$$= \left| \sum_{i,j} S_{ij} e_i e_j \right|^{n/2} \Big/ \prod_1^{n+1} S_{ii}^{1/2} \tag{2}$$

for $n = 1, 2, \cdots$, where R_{ij} is the cofactor of r_{ij} in the $(n+1) \times (n+1)$ matrix $r = (r_{ij})$ and S_{ij} is the cofactor of s_{ij} in $s = (s_{ij})$. Next we determine the largest simplex with given d values and the smallest simplex containing Q with given e values. We find

$$n!\, V_{\max} = \theta^{-1/2} \prod_1^{n+1} (\theta + d_i^2)^{1/2}, \quad r'_{ij} = -\frac{\theta}{d_i d_j}\,, \tag{3}$$