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# SEQUENCES OF CONTRACTIVE MAPS AND FIXED POINTS

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Dedicated to Apollo XI

**Let $(X, \rho_0)$ be a metric space and, for each $n = 0, 1, 2, \cdots$, let $f_n: X \to X$ be a function with fixed point $a_n$. Assume that each function $f_n$ is contractive with respect to a (possibly) different metric $\rho_n$, where each $\rho_n$ is equivalent to $\rho_0$. This paper is concerned with the behavior of the sequence $\{a_n\}_{n=1}^{\infty}$ when $\{f_n\}_{n=1}^{\infty}$ converges pointwise to $f_0$.**

In §1 an example of a compact space $(X, \rho_0)$ is given such that, though $\{\rho_n\}_{n=1}^{\infty}$ converges pointwise to $\rho_0$, $\{a_n\}_{n=1}^{\infty}$ converges, and $f_n$ has $\rho_n$-Lipschitz constant $1/2$, $\{a_n\}_{n=1}^{\infty}$ does not converge to $a_0$. In §2 some theorems are proved assuming uniform convergence of $\{\rho_n\}_{n=1}^{\infty}$ to $\rho_0$. The example in §1 shows that none of the results in §2 remains valid if uniform convergence of the metrics is replaced by pointwise convergence. In §3 a fixed point theorem for compact nonempty set-valued contractive mappings is proved and it is shown that the analogous statement for closed and bounded nonempty set-valued contractive mappings is false. It is then indicated how the results of §2 can be extended to compact nonempty set-valued contractive mappings.

Let $(X, \rho)$ be a metric space. A function $f: X \to X$ is said to be a *$\rho$-contraction* if and only if there exists $\lambda$, $0 \leqq \lambda < 1$, such that $\rho(f(x), f(y)) \leqq \lambda\rho(x, y)$ for all $x, y \in X$($\lambda$ is called a *$\rho$-Lipschitz constant* for $f$). A function $f: X \to X$ is said to be *$\rho$-contractive* if and only if $\rho(f(x), f(y)) < \rho(x, y)$ for all $x, y \in X$, $x \neq y$.

The following theorem was proved in [4].

THEOREM A. *Let $(X, \rho)$ be a locally compact metric space, let $f_n: X \to X$ be a $\rho$-contraction with fixed point $a_n$ for each $n = 1, 2, \cdots$, and let $f_0: X \to X$ be a $\rho$-contraction with fixed point $a_0$. If the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to $f_0$, then the sequence $\{a_n\}_{n=1}^{\infty}$ converges to $a_0$.*

In [5] it was shown that closed and bounded nonempty set-valued contraction mappings defined on a complete space have fixed points. Theorem A and other results in [4] were generalized to compact nonempty set-valued contractions.

Throughout this paper two metrics, $d_1$ and $d_2$, for the same set