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AFFINE COMPLEMENTS OF DIVISORS

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Recently Goodman and Hartshorne have considered the question of characterizing those divisors in a complete linear equivalence class whose support has an affine complement. However their characterization is not clearly "linear", and in fact we have to resort to Serre's characterization of affine schemes to prove that, indeed, the condition "the support of an effective divisor has an affine complement" is, in the language of Italian geometry, expressed by linear conditions. In the language of Weil this means that the set of effective divisors, in a complete linear equivalence class, whose supports have affine complements is a linear system. This is our first result. Subsequently we study the intersection of all such affine-complement supports of effective divisors in the multiples of a given linear equivalence class, and prove the following: if the ambient scheme is a surface or a threefold, or if the characteristic of the groundfield is 0, (or assuming that we can resolve singularities!) then a minimal intersection cannot have zero-dimensional components, nor irreducible components of codimension 1, whose associated sheaf of ideals is invertible.

In particular we obtain anew Zariski's result (see [11]) that every complete nonsingular surface is projective, and that the examples of nonsingular, nonprojective threefolds given by Nagata and Hironaka (see [9] and [5]) are optimal, in the sense that no examples can be given of nonsingular, nonprojective threefolds in which the "bad" subsets are either closed points or two-dimensional subschemes.

The notation and terminology we use are, unless otherwise specifically stated, those of [4]. We consider only algebraic schemes, with an arbitrary, algebraically closed ground field k. For the sake of convenience we drop the adjective "algebraic", and speak simply of schemes.

When we refer to, say, Lemma 2.3 without further identification, we mean Lemma 2.3 of the present work, to be found as the third statement of § 2.

1. Let X be a scheme, $\mathscr{L}$ an invertible sheaf over X. A regular section $s \in \Gamma(X, \mathscr{L})$ identifies an exact sequence

$$0 \longrightarrow \mathscr{L}^{-1} \xrightarrow{\theta(s)} \mathscr{O}_X \longrightarrow \mathscr{K} \longrightarrow 0$$

with $\mathrm{Supp}(\mathscr{K}) = \mathrm{Supp}(s) = \{x \in X \mid s(x) \in \mathscr{M}_x\}$, $\mathscr{M}_x$ denoting the uni-