PACIFIC JOURNAL OF MATHEMATICS
Vol. 31, No. 3, 1969

A NOTE ON GROUPS WITH FINITE DUAL SPACES

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If a locally compact group has only a finite number of inequivalent irreducible unitary representations, then one is tempted to conjecture that it is a finite group. The conjecture is known to be true in certain special cases. We present here a proof in case the group satisfies the second axiom of countability.

PROPOSITION 1.1. *If G is an abelian locally compact group having only a finite number of inequivalent irreducible unitary representations, then G is a finite group.*

This follows immediately from the Pontrjagin duality theorem.

PROPOSITION 1.2. *If G is a compact group having only a finite number of inequivalent irreducible unitary representations, then G is a finite group.*

We may deduce a proof of this from the Peter-Weyl theorem, for example, as follows: $L^2(G)$ is the direct sum $\sum_i I_i$ of finite dimensional subspaces $[I_i]$, where, for each i, I_i is a minimal two-sided ideal in $L^2(G)$. Further, there is a one-to-one correspondence between the set $[I_i]$ of these ideals and the set of all equivalence classes of irreducible unitary representations of G. If the latter set is finite, as assumed, then $L^2(G)$ is finite dimensional, and G is necessarily a finite group.

The proof we give here for the second countable case depends on Dixmier's theory of square-integrable representations, which, in turn, depends on some rather technical results concerning Hilbert algebras. It would be desirable, of course, to discover an elementary proof to what appears to be such an elementary theorem. I have devised a fairly elementary proof—"elementary" in the sense that, beyond the notion of Haar measure, the only deep result needed is Kadison's theorem on the algebraic irreducibility of a topologically irreducible $*$-representation of a C^*-algebra. This proof, however, still suffers from being quite long, so I do not include it.

Regarding the situation when G is an arbitrary locally compact group, there is no direct integral theory available in general, and we therefore lose an important tool for moving from hypotheses about the dual space to conclusions, for example, about the regular representation. I can not make headway in resolving this conjecture even