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$\mathscr{L}-2$ SUBSPACES OF GRASSMANN PRODUCT SPACES

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The subspaces of the second order Grassmann product space consisting of products of a fixed irreducible length k and zero are interesting not only for their own sake and their usefulness when determining the structure of linear transformations on the product space into itself which preserve the irreducible length k, but also because they are isomorphic to subspaces of skew-symmetric matrices of fixed rank $2k$. The structure of these subspaces and the corresponding preservers are known for $k=1$, when the underlying field F is algebraically closed. This paper gives a complete characterization of these subspaces when $k=2$ and F is algebraically closed. When F is not algebraically closed, these subspaces can be different.

Let $\mathscr{U}$ be an n-dimensional vector space over an *algebraically closed* field F. Let $\bigwedge^2 \mathscr{U}$ denote the $\binom{n}{2}$-dimensional space spanned by all Grassmann products $x_1 \wedge x_2$, $x_i \in F$. A vector $f \in \bigwedge^2 \mathscr{U}$ is said to have *irreducible length* k if it can be written as a sum of k, and *not less than* k, nonzero pure (decomposable) products in $\bigwedge^2 \mathscr{U}$. Let $\mathscr{L}_k$ denote the set of all vectors of irreducible length k in $\bigwedge^2 \mathscr{U}$, and $f \in \mathscr{L}_k$ if and only if $\mathscr{L}(f) = k$. A subspace of $\bigwedge^2 \mathscr{U}$ whose nonzero members are in $\mathscr{L}_k$ is called an $\mathscr{L}-k$ *subspace*.

An $\mathscr{L}-2$ subspace H is a (1, 1)-*type subspace* if there exist fixed nonzero vectors $x \neq y$ such that each nonzero $f \in H$ can be written $f = x \wedge x_f + y \wedge y_f$. A basis of a (1, 1)-type subspace is called a (1, 1) basis. When $\dim \mathscr{U} = 4$, every $\mathscr{L}$-2 subspace has dimension one ([4], Th. 10).

It is shown here that (i) for $\dim \mathscr{U} = n \geqq 5$, there always exists an $\mathscr{L}-2$ subspace of (1, 1)-type and dimension two; (ii) the 2-dimensional $\mathscr{L}-2$ subspaces are of (1, 1)-type; (iii) every $\mathscr{L}-2$ subspace of dimension at least four is of (1, 1)-type; (iv) the $\mathscr{L}-2$ subspaces have dimension at most $(n-3)$ when $n \geqq 6$; and this maximum dimension is attained. Also the 3-dimensional $\mathscr{L}-2$ subspaces are characterized, and these are the most varied.

From [4], Theorem 5, each $f \in \mathscr{L}_k$ can be uniquely associated with a $2k$-dimensional subspace $[f]$ of $\mathscr{U}$. The pair $\{f_1, f_2\}$ is said to be a P_m-pair in $\mathscr{L}_2$ if $[f_1] + [f_2]$ has dimension m; and the set $\{f_1, \cdots, f_k\}$ in $\mathscr{L}_2$ is pairwise-P_m if each pair is a P_m-pair, for $i \neq j$.

THEOREM 1. *Let* $\dim \mathscr{U} = n \geqq 5$. *Then there always exists a*