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SOME MATRIX FACTORIZATION THEOREMS

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The object of this paper is to make an exhaustive study of the matrix equation

$$C = ABA^{-1}B^{-1} \tag{1}$$

when A, B, and C are normal matrices. We shall specialize these matrices in various ways by requiring that C, A, or B lie in one or more of the well-known subclasses of the class of normal matrices (Hermitian, unitary, real skew symmetric, etc.). We shall also demand from time to time that C commute with A, or B, or both.

In § 2 we present some notation. In § 3, we prove a number of simple lemmas that will be frequently used. In § 4 we discuss (1) when C is normal and A and B are Hermitian. In § 5, we discuss (1) when C is real and normal and A and B are real and symmetric. In § 6 we present one theorem that is used several times in § 7, where we discuss (1) when C is normal, A is Hermitian, and B is unitary. In § 8 we complete a discussion of (1) when A is Hermitian and B unitary Hermitian that is partly presented in §§ 4, 5, and 7. In §§ 4–7 cases are discussed in which C commutes with A or with B, but not with both. In § 9 we analyse the situation when C commutes with both A and B.

Commutators of normal matrices have been investigated by a number of authors: Fan [1], Frobenius [2], Gotô [3], Marcus and Thompson [5], Taussky [7], Tôyama [9], Zassenhaus [10]. The results obtained in this paper will partly overlap results obtained in [5] but will, in the main, complement the results of [5]. Our principal tools are two elegant tricks due to Ky Fan, both of which appear in his paper [1].

As a consequence of our study of commutators of normal matrices, we are able, through use of the polar factorization theorem, to obtain factorization theorems for nonnormal matrices. It is interesting that we can achieve sharper results for real matrices than for nonreal matrices.

All matrices appearing in this paper, except for the zero matrix, are assumed to be nonsingular.

2. Notation and terminology.

The words symmetric, positive definite symmetric, negative definite symmetric, skew symmetric, orthogonal, will imply that the matrix in question possessing the indicated property is a matrix of real numbers. We shall make use