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LOCALLY GALOIS ALGEBRAS

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Separable subalgebras of commutative algebras which (a) are the direct limit of separable subalgebras and (b) have sufficiently many automorphisms are shown to be the fixed rings of groups of automorphisms of the algebra. Necessary and sufficient conditions for an arbitrary subalgebra to be the fixed ring of a group are examined.

Also, we show that every element of every separable algebra over a ring is separable if and only if the ring is von Neumann regular.

Given a commutative ring R and a commutative R-algebra S, Villamayor and Zelinsky [10, 3.1] call S a *weakly Galois* R-algebra if S is finitely generated and projective as an R-module, separable as an R-algebra, and if there is a finite group G of R-algebra automorphisms of S such that the subring S^G of G-invariant elements of S is precisely R. Under these hypotheses they achieve the following generalization of the fundamental theorem of Galois theory: Every separable R-subalgebra of S is the fixed ring of a finite group of automorphisms, and conversely.

Our study here is of infinitely generated algebras. We say S is a *locally Galois* R-algebra if every finite subset of S is contained in a weakly Galois subalgebra of S with the following property: there is a finite set of automorphisms of the subalgebra, each of which extends to an automorphism of S, and the fixed ring of the subalgebra under this subset is R. We prove that if S is a locally Galois R-algebra, every separable R-subalgebra of S is the fixed ring of some group of automorphisms of S. In general, the converse of this is false.

Call an R-algebra *locally separable* if every finite subset of the algebra is contained in a separable R-subalgebra. Under special hypotheses on R, which allow infinitely many idempotents, but are otherwise rather restrictive, we characterize those locally separable subalgebras of a locally Galois R-algebra which are the fixed rings of groups of automorphisms.

A major technique used here, as in [10], is to reduce to the case where R has no nontrivial idempotents via Pierce's theory of the Boolean spectrum [7]: Let $X(R)$ denote the quotient space of Spec (R) with connected components identified to points. The quotient map of Spec (R) to $X(R)$ induces a sheaf on $X(R)$, the direct image of the canonical sheaf of local rings on R. The resulting ringed space