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ULM'S THEOREM FOR ABELIAN GROUPS MODULO BOUNDED GROUPS

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Let $\hat{A}$ be the category of Abelian groups, $\hat{B}$ the class of bounded Abelian groups. It is shown that if G and H are totally projective p-groups[1], then $G \cong H$ in the quotient category $\hat{A}/\hat{B}$ if and only if there exists an integer $k \geqq 0$ such that for all ordinals α and all integers $r \geqq 0$,

$$\sum_{j=k}^{r+k} f_G(\alpha + j) \leqq \sum_{j=0}^{r+2k} f_H(\alpha + j) \text{ and } \sum_{j=k}^{r+k} f_H(\alpha+j) \leqq \sum_{j=0}^{r+2k} f_G(\alpha + j) .$$

This extends a similar result of R. J. Ensey for direct sums of countable reduced p-groups. It is also noted that if G and H are totally projective p-groups, then G is quasi-isomorphic to H if and only if there exists an integer $k \geqq 0$ such that for all integers $n \geqq 0$ and $r \geqq 0$,

$$\sum_{j=k}^{r+k} f_G(n + j) \leqq \sum_{j=0}^{r+2k} f_H(n + j)$$

and

$$\sum_{j=k}^{r+k} f_H(n + j) \leqq \sum_{j=0}^{r+2k} f_G(n + j) , \quad \text{and} \quad f_G(\alpha) = f_H(\alpha)$$

for all $\alpha \geqq \omega$. This extends a similar result of R. S. Pierce and R. A. Beaumont for direct sums of countable reduced p-groups.

Preliminaries. Let $\hat{A}$ be the category of groups and $\hat{B}$ the Serre class of bounded groups. Then $\hat{A}/\hat{B}$ is the quotient category as defined by Grothendieck [5]. The objects $\hat{A}/\hat{B}$ are the objects of $\hat{A}$.

$$\mathrm{Hom}_{\hat{A}/\hat{B}}(G, H) = \varinjlim_{(G', H') \in D} \mathrm{Hom}(G', H/H') ,$$

where $D = \{G', H' \mid G' \subseteqq G, H' \subseteqq H; G/G', H' \in \hat{B}\}$. D is directed by $(G', H') \leqq (G'', H'')$ if and only if $G'' \subseteqq G'$ and $H' \subseteqq H''$. For a thorough discussion of the category $\hat{A}/\hat{B}$, the reader should see either Ensey [4] or E. A. Walker [9]. From Walker's results, it follows that $G \cong H$ in $\hat{A}/\hat{B}$ if and only if there exist subgroups S and A of G, and T and B of H such that $S/A \cong T/B$ and G/S, H/T, A, and B are bounded. Two groups G and H are quasi-isomorphic if there exist isomorphic subgroups S and T of G and H respectively such that G/S and H/T are bounded. Then clearly, quasi-isomorphism implies isomorphism in $\hat{A}/\hat{B}$, and for torsion-free groups, the converse also holds.

[1] From here on, the word group is used to mean Abelian group.