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AN ANALOGUE OF PTOLEMY'S THEOREM AND ITS CONVERSE IN HYPERBOLIC GEOMETRY

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The purpose of this paper is to give a complete answer to the question: what relations between the mutual distances of n ($n \geqq 3$) points in the hyperbolic plane are necessary and sufficient to insure that those points lie on a line, circle, horocycle, or one branch of an equidistant curve, respectively?

In 1912 Kubota [6] proved an analogue of Ptolemy's Theorem in the hyperbolic plane and recently Kurnick and Volenic [7] obtained another analogue. In 1947 Haantjes [4] gave a proof of the hyperbolic analogue of the ptolemaic inequality, and in [5] he developed techniques which give a new proof of Ptolemy's Theorem and its converse in the euclidean plane. In the latter paper it is further stated that these techniques give proofs of an analogue of Ptolemy's Theorem and its converse in the hyperbolic plane. However, Haantjes' analogue of the converse of Ptolemy's Theorem is false, since Kubota [6] has shown that the determinant $|\sinh^2(P_iP_j/2)|$, where $i, j = 1, 2, 3, 4$, vanishes for four points P_1, P_2, P_3, P_4 on a horocycle in the hyperbolic plane. So far as the author knows, Haantjes is the only person who has mentioned an analogue of the converse of Ptolemy's Theorem in the hyperbolic plane.

Relations between the mutual distances of three points are obtained which are necessary and sufficient to insure that those points determine a line, circle, horocycle, or equidistant curve, respectively.

It will be recalled that Ptolemy's Theorem and its converse for the euclidean plane may be stated as follows.

THEOREM (*Ptolemy*). *Four points P_1, P_2, P_3, P_4 of the euclidean plane lie on a circle or line if and only if the determinant*

$C(P_1, P_2, P_3, P_4) = |P_iP_j^2|$ vanishes, where $P_iP_j = P_jP_i$ denotes the distance of the points P_i, P_j, $(i, j = 1, 2, 3, 4)$.

The analogous theorem which will be proved in this paper is the following.

THEOREM. *Four points P_1, P_2, P_3, P_4 of the hyperbolic plane lie on a circle, line, horocycle, or one branch of an equidistant curve if and only if the determinant*