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# REARRANGEMENT INEQUALITIES INVOLVING CONVEX FUNCTIONS

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**Let $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$ be $n$-tuples of non-negative numbers. Then**

$$(1) \qquad \prod_{i=1}^{n} (a_i'+b_i') \leqq \prod_{i=1}^{n} (a_i+b_i') \leqq \prod_{i=1}^{n} (a_i^*+b_i')$$

**and**

$$(2) \qquad \sum_{i=1}^{n} a_i^* b_i' \leqq \sum_{i=1}^{n} a_i b_i' \leqq \sum_{i=1}^{n} a_i' b_i' .$$

$a' = (a_1', \cdots, a_n')$ **and** $a^* = (a_1^*, \cdots, a_n^*)$ **are respectively the rearrangement of** $a$ **in a nondecreasing or nonincreasing order. (1) was recently found by Minc and (2) is well known. In this note we show that these inequalities are special cases of rearrangement inequalities valid for functions having some convex properties.**

Let $x = (x_1, \cdots, x_n)$ be an $n$-tuple of real numbers. We denote by $x^* = (x_1^*, \cdots, x_n^*)$ the $n$-tuple $x$ rearranged in a nonincreasing order $x_1^* \geqq x_2^* \geqq \cdots \geqq x_n^*$, and we denote by $x' = (x_1', \cdots, x_n')$ the same $n$-tuple rearranged in a nondecreasing order $x_1' \leqq x_2' \leqq \cdots \leqq x_n'$.

Recently Minc [2] proved that if $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$ are real $n$-tuples such that $a_i, b_i \geqq 0$, $i = 1, \cdots, n$, then

$$(1) \qquad \prod_{i=1}^{n} (a_i' + b_i') \leqq \prod_{i=1}^{n} (a_i + b_i') \leqq \prod_{i=1}^{n} (a_i^* + b_i') .$$

If $a_i > 0$ and $b_i \geqq 0$, $i = 1, \cdots, n$, then (1) is equivalent to

$$(1)' \qquad \sum_{i=1}^{n} \log\left(1 + \frac{b_i'}{a_i'}\right) \leqq \sum_{i=1}^{n} \log\left(1 + \frac{b_i'}{a_i}\right) \leqq \sum_{i=1}^{n} \log\left(1 + \frac{b_i'}{a_i^*}\right) .$$

(see also [4, Theorem 2] and [5]).

It is well known [1, Th. 368] that if $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$ are real $n$-tuples, then

$$(2) \qquad \sum_{i=1}^{n} a_i^* b_i' \leqq \sum_{i=1}^{n} a_i b_i' \leqq \sum_{i=1}^{n} a_i' b_i' .$$

If $a_i > 0$ and $b_i \geqq 0$, $i = 1, \cdots, n$, then (2) is obviously equivalent to

$$(2)' \qquad \sum_{i=1}^{n} \left(\frac{b_i'}{a_i'}\right) \leqq \sum_{i=1}^{n} \left(\frac{b_i'}{a_i}\right) \leqq \sum_{i=1}^{n} \left(\frac{b_i'}{a_i^*}\right) .$$

In the present note we generalize (1)' and (2)' for more general