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REGULAR BOUNDARY PROBLEMS FOR A FIVE-TERM RECURRENCE RELATION

C. E. BILLIGHEIMER

We consider in this paper boundary problems for the five-term scalar recurrence relation

$$d_n y_{n+2} + c_n y_{n+1} + (b_n - \lambda a_n) y_n + c_{n-1} y_{n-1} + d_{n-2} y_{n-2} = 0 \quad (0 \leqq n \leqq m) \tag{1.1}$$

where the coefficients a_n, b_n, c_n, d_n are real, $a_n, d_n > 0$ and λ is a complex parameter, with boundary conditions of the typical form

$$y_{-2} = y_{-1} = 0 \tag{1.2}$$

and

$$y_{m+1} + k(c_m y_m + d_{m-1} y_{m-1}) = 0, \; y_{m+2} + h y_m = 0 \tag{1.3}$$

for some integer $m \geqq 0$, and real numbers h, k.

We derive oscillation properties, orthogonality relations and associated eigenvector expansion theorems for solutions of (1.1), (1.2), (1.3), and then discuss the solution of boundary problems for the corresponding inhomogeneous recurrence relation in terms of a Green's function.

Atkinson [1] has discussed the connection between two and three term scalar and matrix recurrence relations and Sturm-Liouville differential equations and first order systems of differential equations.

On this basis the five-term recurrence relation here discussed appears as the analogue of a fourth order differential equation or first order system of dimension four.

The self-adjoint second order differential equation for which the fundamental limit-point, limit-circle distinction for the singular boundary problem first given by Weyl [12] plays an important part is discussed in detail for example in the work of Coddington and Levinson [5]. The analogous three-term recurrence relations were studied by Stone [11] in the setting of Hilbert space theory.

The extension of the theory to the case of the general even order differential equation was given by Kodaira [9] Glazman [8] and Everitt [6], [7] who studied also in particular the fourth order case. A fundamental study of the oscillation theory of the fourth order differential equation was made by Leighton and Nehari [10] and Barrett [2], [3].

We discuss in this paper regular boundary problems for the five-term recurrence relation (1.1). In a subsequent paper (Billigheimer