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AN INTERESTING COMBINATORIAL METHOD IN THE THEORY OF LOCALLY FINITE SEMIGROUPS

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Let X be a finite set, X^* the free semigroup (without identity) on X, let M be a finite semigroup, and let φ be an epimorphism of X^* upon M. We give a simple proof of a combinatorial property of the triple (X, φ, M), and exploit this property to get very simple proofs for these two theorems: 1. If φ is an epimorphism of the semigroup S upon the locally finite semigroup T such that $\varphi^{-1}(e)$ is a locally finite subsemigroup of S for each idempotent element e of T, then S is locally finite. 2. Throughout 1, replace "locally finite" by "locally nilpotent".

The method is simple enough, and yet powerful enough, to suggest its applicability in other contexts.

1. Theorem 1 below was first proved by the author in [1] by a circuitous and laborious method. In the present paper it drops out easily from Lemma 2 below, as does Theorem 2, which is new. Lemma 2 was first discovered by J. Justin ([3]) as a generalization of Lemma 1, which is the author's ([2]). The proof given here, however, is new, and is conceptually quite transparent, though apparently nontrivial. Justin has used Lemma 2 in an alternative proof of his generalization of Van der Waerden's Theorem (on Arithmetic Progressions), using Van der Waerden's Theorem in the course of the proof. The author is inclined to believe that a refined or more powerful version of Lemma 2 would yield a proof of Van der Waerden's Theorem itself.

The construction of a sequence "in the regular way", given below, has been formalized by R. Rado in [4].

2. Notation and definitions. The symbol X will always denote a finite set, and X^* denotes the free semigroup without identity on X. Thus X^* is the semigroup of nonempty "words" in the "letters" of the "alphabet" X, with juxtaposition as multiplication. If $w = x_1x_2 \cdots x_k \in X^*$, where the $x_i \in X$, then the *length* of w, denoted by $|w|$, is k. The symbol X^ω denotes the set of sequences on X, regarded as "infinitely long words" in the alphabet X. If $x, y, z \in X^*$ and $s \in X^\omega$, then x, y, and z are each *factors* (x is a *left factor*) of the word xyz and of the sequence $xyzs$.

Let H be an infinite subset of X^*. We indicate now how to construct a sequence $s = a_1a_2 \cdots \in X^\omega$ such that each left factor of s