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PROPERTIES OF FIXED POINT SETS ON DENDRITES

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Every nonempty closed subset of a dendrite can be the fixed point set of a self-map, but in general it cannot be the fixed point set of a map with special properties. Necessary conditions found here for the fixed point sets of homeomorphisms and monotone surjections of dendrites are mainly concerned with the order of the possible fixed points, and extend earlier results by G. E. Schweigert and L. E. Ward, Jr.

1. Introduction. It was proved in [3, 4] that every closed, nonempty subset of the n-ball B^n can be the fixed point set of a self-map of B^n, but that not all such subsets can be the fixed point set of a homeomorphism of B^n. We investigate in this paper related questions for dendrites. The first result (Theorem 3.1) shows that again every closed nonempty subset can be the fixed point set of a self-map of a dendrite.

It is already known that not every closed nonempty subset A of a dendrite D can be the fixed point set of a homeomorphism of D, or even of a monotone surjection of D. Results for homeomorphisms by G. E. Schweigert [5] and generalizations for monotone maps by L. E. Ward, Jr. [7] show that A cannot consist of one end point of D:

THEOREM 1.1. *(Schweigert and Ward). Let f: $D \to D$ be a monotone surjection of a dendrite D which leaves one end point e of D fixed. Then there exists at least one fixed point distinct from e.*

We extend this theorem in several ways. In § 4 we prove more details about the order (see [8, p. 48]) of the possible fixed points if the fixed point set consists of only finitely many points. The theorem by Schweigert and Ward states that the fixed point set of a monotone surjection cannot consist of one end point, i.e., of one point of order one. We show in Theorem 4.1 that it also cannot consist of two points of order two, and in the case of a homeomorphism it cannot consist of three points of order three. But it can consist of n points of order n for all $n > 3$. We further strengthen Theorem 1.1 by proving a restriction on the fixed point different from e: if f is a homeomorphism, then it can be chosen of an order $\neq 2$ (Theorem 4.5). This is no longer true for monotone surjections.

The work by Schweigert and Ward is concerned with fixed point