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SOME NUMBER THEORETIC RESULTS

(In memory of our good friend Leo Moser)

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The paper first establishes the order of magnitude of maximal sets, S, of residues (mod p) so that the sums of different numbers of elements are distinct.

In the second part irrationalities of Lambert Series of the form $\sum f(n)/a_1 \cdots a_n$ are obtained where $f(n) = d(n)$, $\sigma(n)$ or $\varphi(n)$ and the a_i are integers, $a_i \geqq 2$, which satisfy suitable growth conditions.

This note consists of two rather separate topics. In §1 we generalize a topic from combinatorial number theory to get an order of magnitude for the number of elements in a maximal set of residues (mod p) such that sums of different numbers of elements from this set are distinct. We show that the correct order is $cp^{1/3}$ although we are unable to establish the correct value for the constant c.

Section 2 consists of irrationality results on series of the form $\Sigma f(n)/a_1a_2\cdots a_n$ where $f(n)$ is one of the number theoretic functions $d(n)$, $\sigma(n)$ or $\varphi(n)$ and a_n are integers $\geqq 2$. For $f(n) = d(n)$ it suffices that the a_n are monotonic while for $\sigma(n)$ and $\varphi(n)$ we needed additional conditions on their rates of growth.

1. Maximal sets in a cyclic group of prime order for which subsets of different orders have different sums. In an earlier paper [4] one of us has given a partial answer to the question:

What is the maximal number $n = f(x)$ of integers $a_1, \cdots, a_n$ so that $0 < a_1 < a_2 < \cdots < a_n \leqq x$ and so that

$$a_{i_1} + \cdots + a_{i_s} = a_{j_1} + \cdots + a_{j_t} \text{ for some } \begin{array}{l} 1 \leqq i_1 < \cdots < i_s \leqq n \\ 1 \leqq j_1 < \cdots < j_t \leqq n \end{array}$$

implies $s = t$? it is conjectured that the maximal set is obtained (loosely speaking) by taking the top $2\sqrt{x}$ integers of the interval $(1, x)$. We were indeed able to prove that $f(x) < c\sqrt{x}$ for suitable c (for example $4/\sqrt{3}$) by using the fact that a set of n positive integers has a minimal set of distinct sums of t-tuples $(1 \leqq t \leqq n)$ if it is in arithmetic progression.

It is natural to pose the analogous question for elements of cyclic groups of prime order, as was done at the Number Theory Symposium in Stony Brook [5]. Here again we may conjecture that a maximal set of residues (mod p) is attained by taking a set of consecutive residues, this time not at the upper end but near $p^{2/3}$.