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SOLVABLE AND SUPERSOLVABLE GROUPS IN WHICH EVERY ELEMENT IS CONJUGATE TO ITS INVERSE

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Let $\mathfrak{S}$ be the class of finite groups in which every element is conjugate to its inverse. In the first section of this paper we investigate solvable groups in $\mathfrak{S}$: in particular we show that if $G \in \mathfrak{S}$ and G is solvable then the Carter subgroup of G is a Sylow 2-subgroup and we show that any finite solvable group may be embedded in a solvable group in $\mathfrak{S}$. In the second section the main theorem reduces the study of supersolvable groups in $\mathfrak{S}$ to the study of groups in $\mathfrak{S}$ whose orders have the form $2^{\alpha}p^{\beta}$, p an odd prime.

NOTATION. The notation here will be as in [1] with the addition of the notation $G = XY$ to mean G is a split extension of Y by X. Also, $F(G)$ will denote the Fitting subgroup of G and $\Phi(G)$ the Frattini subgroup of G. We will denote the maximal normal subgroup of G of odd order by $O_{2'}(G)$. Further, $\mathrm{Hol}\,(G)$ will denote the split extension of G by its automorphism group.

If K and T are subgroups of G we will call K a T-group if $T \leqq N_G(K)$ and we say K is a T-indecomposable T-group if $K = K_1 \times K_2$, where K_1 and K_2 are T-groups, implies $K_1 = \langle 1 \rangle$ or $K_2 = \langle 1 \rangle$.

1. Burnside [2] proved that if P is a Sylow p-subgroup of the finite group G and if X and Y are P-invariant subsets of P which are not conjugate in $N_G(P)$ then they are not conjugate in G. Using Burnside's method one may prove a similar fact about the Carter subgroups. The proof is easy and we omit it.

LEMMA 1.1. *Let C be a Carter subgroup of the solvable group G and let A and B be subsets of C, both normal in C. If $A \neq B$ then A and B are not conjugate in G.*

THEOREM 1.1. *If G is a solvable group in $\mathfrak{S}$ then a Carter subgroup of G is a Sylow 2-subgroup of G.*

Proof. Let C be a Carter subgroup of G. If C has a nonidentity element of odd order then C has a nonidentity central element g of odd order, since C is nilpotent. Then with $A = \{g\}$ and $B = \{g^{-1}\}$ the hypotheses of Lemma 1.1 are satisfied and, since $A \neq B$, g and g^{-1} are not conjugate in G, contradicting our supposition that $G \in \mathfrak{S}$.