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# ON SPECTRAL THEORY AND SCATTERING FOR ELLIPTIC OPERATORS WITH SINGULAR POTENTIALS

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**General conditions have been found which imply that the perturbation $A + q$ of an elliptic differential operator $A$ by a singular potential term $q(x)$ has a closed extension $B$ in $L^2(R^n)$ having the same essential spectrum as $A$. The purpose of this paper is to sharpen the known results slightly and to estimate the characteristic numbers of the operator $(A + \lambda)^p - (B + \lambda)^p$. Under an appropriate assumption on $q(x)$, this operator is shown to be of trace class for large $p$. In the self-adjoint case it follows then from results of Kato that wave operators for the pair $(A, B)$ exist and that the absolutely continuous parts of these operators are unitarily equivalent.**

Let $r$ be a positive integer and let

$$A(x, D) = \sum_{|\alpha| \leqq r, |\beta| \leqq r} D^\alpha(a_{\alpha\beta}(x)D^\beta)$$

be a differential operator of order $m = 2r$. Here

$$x \in R^n,\ \alpha = (\alpha_1, \cdots, \alpha_n),\ |\alpha| = \sum \alpha_j\ ,$$

and $D^\alpha = (-i)^{|\alpha|} \prod (\partial/\partial x_j)^{\alpha_j}$. We assume throughout that $a_{\alpha\beta}$ has continuous derivatives of order $\leqq \max\{|\alpha|, |\beta|\}$, and the derivatives are uniformly bounded. For $|\alpha| = |\beta| = r$ we assume $a_{\alpha\beta}$ uniformly continuous. Finally, $A(x, D)$ is uniformly strongly elliptic: there is a constant $a_0 > 0$ such that

$$Re\Big(\sum_{|\alpha|=|\beta|=r} a_{\alpha\beta}(x)\xi^\alpha\xi^\beta\Big) \geqq a_0 |\xi|^m$$

for all $x \in R^n$, $\xi \in R^n$.

Let $A_0$ be the restriction of $A(x, D)$ to $\mathscr{D}$, the smooth functions with compact support. Let $A$ be the closure of $A_0$ in $L^2 = L^2(R^n)$. Various conditions have been given on a potential term $q(x)$ such that $A + q$ have a closed extension $B$ with the same essential spectrum as $A$, either generally or in the particular case $A = -\Delta$; [1], [2], [4], [6]. The most general result of this sort seems to be that of Schechter [8], [9]. We shall sharpen Schechter's result and then investigate the characteristic numbers of $(A + \lambda)^{-p} - (B + \lambda)^{-p}$.

For $\mu > -n$ and $x \neq 0$, set $\omega_\mu(x) = |x|^\mu$ if $\mu < 0$,