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THE FUNCTIONS OF BOUNDED INDEX AS A SUBSPACE OF A SPACE OF ENTIRE FUNCTIONS

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Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be entire functions. Define $d(f, g) = \text{Sup}\,\{|a_0 - b_0|, (|a_n - b_n|)^{1/n}\ n = 1, 2, \cdots\}$. It is the purpose of this note to show that, in the topology generated by d, the entire functions of bounded index, B, are of the first category.

Further, for Γ, the corresponding space of all entire functions, and $B_n = \{f \in B \mid \text{the index of } f \text{ is} \leqq n\}$ is shown that $B - B_n$ is dense in Γ for any nonnegative integer n. It is also shown that $\Gamma - B$ is dense in Γ. (For definition and main results see [2], [3].)

LEMMA 1. *For any $f \in \Gamma$, $N \geqq 0$, and $\varepsilon > 0$ there exists a $\delta > 0$ such that if $g \in \Gamma$ and $d(f, g) < \delta$ then $d(f^{(k)}, g^{(k)}) < \varepsilon$ for $k = 0, 1, \cdots, N$.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \Gamma$, $N \geqq 0$, and $\varepsilon > 0$ be given. Let

$$T > \text{Sup}\left\{\left(\frac{(n+k)!}{n!}\right)^{1/n} \mid n = 1, 2, \cdots \text{ and } k = 0, 1, \cdots, N.\right\}.$$

It is straightforward to verify that if $g(z) = \sum_{n=0}^{\infty} b_n z^n \in \Gamma$ and $d(f, g) < \dfrac{\varepsilon}{T + \varepsilon}$ then

$$\begin{aligned} &d(f^{(k)}, g^{(k)}) \\ &\quad = \text{Sup}\left\{k!\,|a_k - b_k|, \left(\frac{n+k)!}{n!}\,|a_{n+k} - b_{n+k}|\right)^{1/n} n = 1, 2, \cdots\right\} \\ &\quad < T \cdot \frac{\varepsilon}{T + \varepsilon} < \varepsilon \text{ for } k = 0, 1, \cdots, N. \end{aligned}$$

REMARK. If $f \in \Gamma - B$ then f is said to be of unbounded index and the index of $f = \infty$.

LEMMA 2. *If n is a nonnegative integer and f is of index $> n$* (bounded or unbounded) *then there exists a $\delta > 0$ such that if $g \in \Gamma$ and $d(f, g) < \delta$ then $g \in \Gamma - B_n$.*

Proof. Let n be given such that $n \geqq 0$. Let $f \in \Gamma$ be given such that the index of f (bounded or unbounded) is $> n$. Let k be