PACIFIC JOURNAL OF MATHEMATICS
Vol. 37, No. 2, 1971

INVERSION OF THE HANKEL POTENTIAL TRANSFORM

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We consider the Hankel potential transform

$$f(x) = \int_0^\infty \frac{t}{(x^2+t^2)^{\nu+1}}\phi(t)d\mu(t),\ \nu > 0\ , \tag{1.1}$$

where

$$d\mu(x) = \frac{1}{2^{\nu-1/2}\Gamma(\nu+1/2)}x^{2\nu}dx\ . \tag{1.2}$$

This transform is intimately related to the Hankel transform. Indeed, its kernel is the Hankel transform

$$\frac{t}{(x^2+t^2)^{\nu+1}} = \frac{\sqrt{\pi}}{2^{\nu+1/2}\Gamma(\nu+1)}\int_0^\infty \mathscr{J}(xy)e^{-tx}d\mu(y)\ , \tag{1.3}$$

where

$$\mathscr{J}(z) = 2^{\nu-1/2}\Gamma\left(\nu+\frac{1}{2}\right)z^{-\nu+1/2}J_{\nu-1/2}(z)\ ,$$

$J_\gamma(z)$ **being the ordinary Bessel function of order** γ**. Our object is to develop an inversion theory for (1.1) and to exploit the relationship of (1.1) to the Hankel transform.**

When $\nu = 0$, some of our results reduce, modulo a constant, to those of D. V. Widder in [6], a paper on which the present work is closely based. In [7], Widder derived an inversion theory for the general transform

$$f(x) = \int_0^\infty K\left(\frac{x}{t}\right)\frac{\phi(t)}{t}dt\ , \tag{1.4}$$

which includes his result in [6] as a special case; however, the transform (1.1) for $\nu > 0$ is not covered by that development.

2. Preliminaries. The differential operator L_x which is to effect the desired inversion of (1.1) is defined as follows:

$$L_x[f(x)] = \lim_{n\to\infty} L_{n,x}[f(x)]\ , \tag{2.1}$$

where, with D denoting differentiation with respect to x,

$$L_{1,x}[f] = \frac{\sqrt{\pi}\,\Gamma(2\nu+1)}{2^{5/2+\nu}[\Gamma(2+\nu/2)]^2x^{1+2\nu}}Dx^{5+2\nu}D\frac{1}{x}D[f]\ .$$

$$\begin{aligned} L_{n,x}[f] = {} & \frac{(-1)^{n+1}\sqrt{\pi}\,\Gamma(2\nu+1)}{2^{2n+\nu+1/2}[\Gamma(n+\nu/2+1)]^2x^{2n+2\nu-1}}Dx^{4n+2\nu+1} \\ & \times\left[\prod_{k=1}^{n-1}D\frac{1}{x^{4k+2\nu+1}}Dx^{4k+2\nu+1}\right]D\frac{1}{x}D[f],\ n = 2, 3, \cdots \end{aligned} \tag{2.2}$$