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ON POINT-FREE PARALLELISM AND WILCOX LATTICES

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A Wilcox lattice L is constructed from a complemented modular lattice $\varLambda$, by deleting nonzero elements of some ideal of $\varLambda$ and by introducing in the remains L the same order as $\varLambda$. The lattice $\varLambda$ is called the modular extension of L. Using the theory of parallelism in atomistic lattices, it was proved that any affine matroid lattice is an atomistic Wilcox lattice, that is, an existence theorem of the modular extension in the atomistic case. The main purpose of this paper is to extend this result to the general case, by the use of arguments on point-free parallelism.

A matroid lattice is an upper continuous atmistic lattice with the covering property. In the book [2] of Dubreil-Jacotin, Leisieur and Croisot, a generalized affine geometry is defined as a weakly modular matroid lattice of length $\geqq 4$, satisfying the Euclid's weak parallel axiom. This lattice is called an affine matroid lattice in [4] and [5]. In [2], pp. 311–314, it is proved that any affine matroid lattice has the modular extension and hence it is a Wilcox lattice. One can see that the key theorems in the proof of this result are the transitivity theorem of parallelism and theorems on the incomplete elements.

In this paper, we consider a sectionally semicomplemented lattice L with some join-dense set of modular elements (see § 1). This is a generalization of an atomistic lattice. Instead of the parallelism in matroid lattices, we use the point-free parallelism introduced by F. Maeda [6]. In § 2, we give some fundamental results on point-free parallelism. In § 3, we introduce three axioms (P 1), (P 2) and (P 3) on point-free parallelism in L, which are satisfied if L is a Wilcox lattice. In the subsequent three sections, we assume that L is weakly modular, left complemented and of length $\geqq 4$, and that L satisfies (P 1) and (P 2). The main result in § 4 is the transitivity of point-free parallelism. In § 5, we define the parallel images of incomplete elements which generalize those defined in [5], § 4. In § 6, adding the axiom (P 3) in a special case, we construct the modular extension of L and we get two main theorems 6.1 and 6.2.

1. Preliminaries.

In a lattice L, we write $(a, b)M$ when

$$(c \vee a) \wedge b = c \vee (a \wedge b) \quad \text{for} \quad c \leqq b .$$

An element $a \in L$ is called a *modular* element when $(x, a)M$ for every